SOME COEFFICIENT ESTIMATES FOR H^p FUNCTIONS

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Abstract. We find the maximum modulus of the n-th Taylor coefficient c_n of a function in the unit ball of H^p, $1 \le p \le \infty$, provided that c_0 is fixed, and identify the corresponding extremal functions.

1. Introduction

We are interested in finding the maximum Taylor coefficient of a function in the unit ball of H^p, $1 \le p \le \infty$, whose value c at the origin is fixed and in identifying the corresponding extremal functions.

The motivation for this problem comes in part from the Hausdorff-Young inequality and the Bohr phenomenon. The Hausdorff-Young inequality states that if $f \in L^p = L^p(\mathbb{T})$, where $\mathbb{T} = \{z \in \mathbb{C} : |z| = 1\}$, then

$$\|f\|_{L^p} \ge \|\{\hat{f}(n)\}\|_{l^q}$$

for $1 \le p \le 2$, and q such that $\frac{1}{p} + \frac{1}{q} = 1$. If $p = 2$, then $q = 2$ and we have equality; this is Parseval's formula. If $p > 2$, the inequality is known to fail. In particular, if $p = \infty$, it is obviously not the case that

$$\|f\|_\infty \ge \sum_{n=-\infty}^{\infty} |\hat{f}(n)|.$$

However, for $f \in H^\infty$ and $r = \frac{1}{3}$,

$$\|f\|_\infty \ge \sum_{n=0}^{\infty} |\hat{f}(n)| r^n.$$

This is a classical theorem of H. Bohr ([3]) and can be shown from F. Wiener's estimates [2]. One can ask whether such a phenomenon exists

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for H^p, $2 < p < \infty$. Namely, does there exist $0 < r < 1$ such that

$$\|f\|_p \geq (\sum_{n=0}^{\infty} |\hat{f}(n)|^q r^n)^{\frac{1}{q}}?$$

Our estimates of the Taylor coefficients of H^p functions show that no such Bohr phenomenon holds when $2 < p < \infty$.

Finally, our results give a new proof of a conjecture by Hummel, Scheinberg, and Zalcman (see [5], p. 189) related to the Krzyz problem for H^p functions in the case that $n = 1$. The conjecture states that for $n \geq 1$

$$\sup\{|\hat{f}(n)| : f \in H^p, ||f||_p \leq 1, f \text{ zero-free }\} = (\frac{2}{e})^{\frac{1}{q}}$$

and gives the form of the corresponding extremal function.

2. The Main Problem and Known Cases

Problem. *Given* $1 \leq p \leq \infty$, $0 \leq c \leq 1$, *and* $n \geq 1$, *find*

$$M_p(n, c) = \max\{|\hat{f}(n)| : f \in H^p, \|f\|_p \leq 1, |f(0)| = c\},$$

where $f(z) = \sum_{j=0}^{\infty} \hat{f}(j) z^j$, *and identify the extemal functions.*

Notice that if f is extremal for $M_p(n, c)$, then so is $e^{i\alpha} f(e^{i\beta} z)$. We will therefore consider extremal f so that $f(0) = c$ and $\hat{f}(n) \geq 0$. This problem is known for certain cases. If $p = 2$, by Parseval's identity, it is easy to see that $M_2(n, c) = \sqrt{1 - c^2}$ and the corresponding unique extremal function is $f_{ext}(z) = c + \sqrt{1 - c^2} z^n$. When $c = 0$, for all $1 \leq p \leq \infty$, $M_p(n, 0) = 1$ and $f_{ext}(z) = z^n$. When $c = 1$, $M_p(n, 1) = 0$, and $f_{ext}(z) = 1$. If $p = \infty$, $M_\infty(n, c) = 1 - c^2$ (see [2]) and the corresponding unique extremal function is $f_{ext}(z) = \frac{c + z^n}{1 + c z^n}$.

In addition, it is easily seen that for all $p \geq 1$, $M_p(n, c) = M_p(1, c)$. For suppose f is a solution to the given extremal problem. Consider

$$\begin{aligned} \tilde{f}(z) &= \frac{1}{n} \sum_{k=0}^{n-1} f(e^{\frac{2\pi k}{n} i} z^{\frac{1}{n}}) \\ &= \sum_{j=0}^{\infty} \hat{f}(jn) z^j. \end{aligned}$$

Then $\tilde{f} \in H^p$, $\|\tilde{f}\|_p \leq \|f\|_p$, $\tilde{f}(0) = f(0)$, and $\tilde{f}'(0) = \hat{f}(n)$. Therefore the extremal values of $\hat{f}(n)$ are the same as the ones calculated in the case when $n = 1$.

All other cases are unknown. Therefore we consider the extremal problem stated above for values of p such that $1 \leq p < \infty$ $(p \neq 2)$

and c such that $0 < c < 1$. In the following sections, we consider only such values of p and c. In proving the main result, we prove some intermediate theorems which are of independent interest.

3. Statement of the Main Results

Theorem 3.1. *If* $2^{-\frac{1}{p}} \leq c \leq 1$, *then*

$$M_p(n,c) = \frac{2}{p}c^{1-\frac{p}{2}}\sqrt{1-c^p}$$

and the corresponding extremal function is

$$f(z) = (c^{\frac{p}{2}} + \sqrt{1-c^p}z^n)^{\frac{2}{p}}.$$

Theorem 3.2. *If* $0 < c < 2^{-\frac{1}{p}}$, *then the zero-free function* f *such that* $\|f\|_p \leq 1$ *and* $|f(0)| = c$ *that maximizes* $|f'(0)|$ *is*

$$f(z) = 2^{-\frac{1}{p}}(1+z)^{\frac{2}{p}}(2^{\frac{1}{p}}c)^{\frac{1-z}{1+z}}$$

and

$$f'(0) = c(\frac{2}{p} + \log\frac{1}{2^{\frac{2}{p}}c^2}).$$

Theorem 3.3. *If* $0 < c < 2^{-\frac{1}{p}}$, *then*

$$M_p(n,c) = (\frac{2}{p} - 1)cv + \frac{c}{v}$$

and the corresponding extremal function is

$$f(z) = \frac{c}{v}(1+vz^n)^{\frac{2}{p}-1}(v+z^n)$$

where v *is the unique root (* $0 < v \leq 1$*) of* $v^p - c^p = c^p v^2$. *In particular, for* $p = 1$ *and* $0 < c < \frac{1}{2}$, $M_1(n,c) = 1$ *and* $f(z) = c + z^n + cz^{2n}$.

4. Intermediate Results

In the following theorems, we will take $0 < c < 1$.

Proposition 4.1. *The inner function* G *that maximizes* $ReG'(0)$ *if* $G(0) = c$ *is*

$$G(z) = \frac{c+z}{1+cz}$$

and

$$G'(0) = 1 - c^2.$$

Proof. This follows directly from the H^∞ case, since the solution to the maximal problem in that situation is in fact the above inner function. □

Proposition 4.2. *The singular function $S(z)$ that maximizes $ReS'(0)$ if $S(0) = c$ is*

$$S(z) = c^{\frac{1-z}{1+z}}$$

and

$$S'(0) = 2c \log \frac{1}{c}.$$

Proof. Since S is a singular function, there exists a singular positive measure μ such that

$$S(z) = \exp(-\int_{-\pi}^{\pi} \frac{e^{it}+z}{e^{it}-z} d\mu(t)).$$

Therefore

$$S'(z) = S(z)(-\int_{-\pi}^{\pi} \frac{2e^{it}}{(e^{it}-z)^2} d\mu(t)).$$

In particular,

$$S'(0) = c(-\int_{-\pi}^{\pi} 2e^{-it} d\mu(t)).$$

Therefore

$$\begin{aligned} ReS'(0) &= c(-\int_{-\pi}^{\pi} 2\cos t d\mu(t)) \\ &\leq 2c \int_{-\pi}^{\pi} d\mu(t) \\ &= 2c \log \frac{1}{c} \end{aligned}$$

Notice that this maximum is in fact attained when μ has full mass at $t = \pi$ and thus

$$S(z) = c^{\frac{1-z}{1+z}}.$$

□

Proposition 4.3. *If $0 < c < 2^{-\frac{1}{p}}$, then there is no outer function f that maximizes $Ref'(0)$ under the restrictions $\|f\|_p \leq 1$ and $f(0) = c$.*

Proof. For simplicity, we will consider $p = 1$, the proof for $p > 1$ being similar. Let $f \in H^1$ be an outer function such that $\|f\|_1 \leq 1$ and $f(0) = c$. Then

$$f(z) = \exp(\frac{1}{2\pi} \int_{-\pi}^{\pi} \frac{e^{it}+z}{e^{it}-z} \varphi(t) dt)$$

where $\varphi(t) = \log|f(e^{it})|$. Taking derivatives and real parts, we can conclude that

$$Ref'(0) = 2c \frac{1}{2\pi} \int_{-\pi}^{\pi} \varphi(t) \cos t dt.$$

We now have a variational problem, where we need to find

$$\max \frac{1}{2\pi}\int_{-\pi}^{\pi}\varphi(t)\cos t dt$$

under the constraints $\frac{1}{2\pi}\int_{-\pi}^{\pi} e^{\varphi(t)}dt = 1$ and $\frac{1}{2\pi}\int_{-\pi}^{\pi}\varphi(t)dt = \log c < 0$.

If this maximum equals μ and is attained when $\varphi(t) = \varphi_0(t)$, then φ_0 also solves the following dual variational problem: find

$$\min \frac{1}{2\pi}\int_{-\pi}^{\pi} e^{\varphi(t)}dt$$

under the constraints $\frac{1}{2\pi}\int_{-\pi}^{\pi}\varphi(t)\cos t dt = \mu$ and $\frac{1}{2\pi}\int_{-\pi}^{\pi}\varphi(t)dt = \log c$, because the above minimum is then equal to 1. To see this, suppose that for some $\varphi(t) = \varphi_1(t)$ satisying the constraints of the dual problem $\frac{1}{2\pi}\int_{-\pi}^{\pi} e^{\varphi_1(t)}dt < 1$. Then there is some $s > 0$ such that the function $\varphi_2(t) = \varphi_1(t) + s\cos t$ satisfies $\frac{1}{2\pi}\int_{-\pi}^{\pi}\varphi_2(t)\cos t dt = \mu + \frac{s}{2}$, $\frac{1}{2\pi}\int_{-\pi}^{\pi}\varphi_2(t)dt = \log c$, and $\frac{1}{2\pi}\int_{-\pi}^{\pi} e^{\varphi_2(t)}dt = 1$, contrary to the assumed maximality of μ in the original variational problem.

The constraints in the dual problem say that in the real Fourier series of φ the two coefficients are fixed, namely the constant term and the coefficient of $\cos t$. By a standard variational argument we obtain that, whenever a bounded function $h(t)$ satisfies

$$\int_{-\pi}^{\pi} h(t)dt = \int_{-\pi}^{\pi} h(t)\cos t dt = 0,$$

the following must hold:

$$\Big(\frac{d}{ds}\int_{-\pi}^{\pi} e^{\varphi_0(t)+sh(t)}dt\Big)_{s=0} = \int_{-\pi}^{\pi} e^{\varphi_0(t)}h(t)dt = 0.$$

In particular, this is true for $h(t) = \sin nt$ $(n > 0)$ and $h(t) = \cos nt$ $(n \neq 0, 1)$, which implies that $e^{\varphi_0(t)} = A + B\cos t$, $\varphi_0(t) = \log(A + B\cos t)$, where A and B are constants, and $\varphi_0(t)$ is the extremal function for the original variational problem.

Since $\frac{1}{2\pi}\int_{-\pi}^{\pi} e^{\varphi_0(t)}dt = 1$, $A = 1$. Therefore $0 \le B < 1$. Consider now the function

$$k(\gamma) = \frac{1}{2\pi}\int_{-\pi}^{\pi}\log(1 + \gamma\cos t)dt$$

for $0 \le \gamma \le 1$. Notice that $k(B) = \log c$. Also $k'(\gamma) < 0$, $k(0) = 0$ and $k(1) = -\log 2$. This forces $c \ge \frac{1}{2}$, which is a contradiction. Therefore no such constant B exists, and there is no solution to the extremal problem. $\square$

5. Proofs of the Main Results

We begin by considering the main problem for functions whose value at the origin is not close to 0 as stated in Theorem 3.1.

Proof. Let $2^{-\frac{1}{p}} \leq c < 1$, $n = 1$, and consider $f \in H^p$ such that $f(0) = c$ and $\|f\|_p \leq 1$. Write $f(z) = B(z)F(z)$, where B is a Blaschke product and where F has no zeros. Let $B(0) = v > 0$ and $F(0) = u > 0$. Note that $uv = c$. Since $B \in H^\infty$ and $\|B\|_\infty = 1$, F. Wiener's estimates ([2]) show that

$$|B'(0)| \leq 1 - v^2.$$

Note that $F \in H^p$, $\|F\|_p = \|f\|_p = 1$, and F has no zeros. Define $G(z) = F(z)^{\frac{p}{2}}$. Then $\|G\|_2^2 = \|F\|_p^p = 1$ and $G(0) = u^{\frac{p}{2}}$. Therefore, by the case $p = 2$,

$$|G'(0)| \leq \sqrt{1 - u^p}.$$

Writing $F(z) = G(z)^{\frac{2}{p}}$, we get

$$\begin{aligned} |F'(0)| &\leq \frac{2}{p}(u^{\frac{p}{2}})^{\frac{2}{p}-1}\sqrt{1-u^p} \\ &= \frac{2}{p}u^{1-\frac{p}{2}}\sqrt{1-u^p} \end{aligned}$$

Putting the two estimates above together gives

$$\begin{aligned} |f'(0)| &= |B'(0)F(0) + B(0)F'(0)| \\ &\leq (1-v^2)u + v\frac{2}{p}u^{1-\frac{p}{2}}\sqrt{1-u^p} \\ &= c(\frac{1}{v} - v) + \frac{2}{p}c^{1-\frac{p}{2}}\sqrt{v^p - c^p} \end{aligned}$$

Therefore, define

$$\varphi(v) = c(\frac{1}{v} - v) + \frac{2}{p}c^{1-\frac{p}{2}}\sqrt{v^p - c^p}$$

and find $\max \varphi(v)$ for $c \leq v \leq 1$. Note that

$$\begin{aligned} \varphi'(v) &= c(-\frac{1}{v^2} - 1) + \frac{2}{p}c^{1-\frac{p}{2}}\frac{1}{2}\frac{1}{\sqrt{v^p - c^p}}pv^{p-1} \\ &= -c(1 + \frac{1}{v^2}) + c^{1-\frac{p}{2}}\frac{v^{p-1}}{\sqrt{v^p - c^p}} \end{aligned}$$

Therefore,

$$\begin{aligned}\varphi'(v) \geq 0 \quad &\Leftrightarrow \quad c^{-\frac{p}{2}} \frac{v^{p-1}}{\sqrt{v^p - c^p}} \geq 1 + \frac{1}{v^2} \\ &\Leftrightarrow \quad (1+v^2)^2 (c^p)^2 - v^p (1+v^2)^2 c^p + v^{2(p+1)} \geq 0 \\ &\Leftrightarrow \quad (c^p - \frac{v^p}{1+v^2})(c^p - \frac{v^{2+p}}{1+v^2}) \geq 0\end{aligned}$$

When $c \geq 2^{-\frac{1}{p}}$, $\varphi'(v) \geq 0$ and the maximum of $\varphi(v)$ is obtained at $v = 1$:

$$\varphi(1) = \frac{2}{p} c^{1-\frac{p}{2}} \sqrt{1 - c^p}.$$

In that case, the function

$$f(z) = (c^{\frac{p}{2}} + \sqrt{1 - c^p} z)^{\frac{2}{p}}$$

is an element of H^p with norm 1 such that $f(0) = c$ and $f'(0) = \varphi(1)$.

When $n > 1$, use the function $\tilde{f}$ described in Section 2. Since $f(z) = \tilde{f}(z^n)$, we obtain the extremal function

$$f(z) = (c^{\frac{p}{2}} + \sqrt{1 - c^p} z^n)^{\frac{2}{p}}$$

with the same maximal n-th Taylor coefficient as in the case $n = 1$. □

Notice that f is a zero-free function, and therefore Theorem 3.1 also solves the extremal problem for zero-free H^p functions whose value at the origin is not too close to 0. Let us now consider zero-free functions in H^p whose value at the origin are small, as stated in Theorem 3.2.

Proof. Let $0 < c < 2^{-\frac{1}{p}}$ and let $f \in H^p$ be a non-zero function such that $f(0) = c$ and $\|f\|_p \leq 1$ for which $|f'(0)|$ is maximal. Write $f(z) = S(z)F(z)$ where S is a singular function and F is an outer function. Writing $S(0) = u$ and $F(0) = v$, notice that by Proposition 4.3, $v \geq 2^{-\frac{1}{p}}$. Using the estimates given by Proposition 4.2 and Theorem 3.1, we get that

$$\begin{aligned}|f'(0)| \quad &\leq \quad v 2u \log \frac{1}{u} + u \frac{2}{p} v^{1-\frac{p}{2}} \sqrt{1 - v^p} \\ &= \quad 2c \log \frac{v}{c} + \frac{2c}{p} \sqrt{\frac{1}{v^p} - 1} \\ &= \quad \varphi(v)\end{aligned}$$

One can easily show that $\varphi(v)$ is decreasing on $[2^{-\frac{1}{p}}, 1]$ and therefore attains its maximum at $v = 2^{-\frac{1}{p}}$. Therefore $u = c 2^{\frac{1}{p}}$, and the function

$$f(z) = (2^{\frac{1}{p}} c)^{\frac{1-z}{1+z}} 2^{-\frac{1}{p}} (1+z)^{\frac{2}{p}}$$

is a zero-free function such that $f(0) = c$, $\|f\|_p = 1$ and

$$f'(0) = \varphi(2^{-\frac{1}{p}}) = c(\frac{2}{p} + \log \frac{1}{2^{\frac{2}{p}} c^2}).$$

□

We now consider functions in H^p that can have zeros and whose value at the origin is small, as stated in Theorem 3.3.

Proof. Consider the case $n = 1$ and let $f \in H^p$ be such that $\|f\|_p \leq 1$ and $f(0) = c$. Write $f(z) = B(z)F(z)$ where B is a Blaschke product with $B(0) = v > 0$, and F is zero-free with $F(0) = u$.

Suppose first that $u \geq 2^{-\frac{1}{p}}$. Then $c \leq v \leq 2^{\frac{1}{p}} c$, so by the proof of Theorem 3.1

$$\begin{aligned} |f'(0)| &\leq c(\frac{1}{v} - v) + \frac{2}{p} c^{1-\frac{p}{2}} \sqrt{v^p - c^p} \\ &= \varphi(v) \end{aligned}$$

and

$$\varphi'(v) \geq 0 \iff (c^p - \frac{v^p}{1+v^2})(c^p - \frac{v^{2+p}}{1+v^2}) \geq 0.$$

If $c < 2^{-\frac{1}{p}}$ then $\varphi'(v) = 0$ has two solutions v_1 and v_2 in $[c, 1]$ where v_1 is the solution to $v^p - c^p = c^p v^2$ and v_2 is the solution to $v^{2+p} - c^p = c^p v^2$. It is not hard to show that

$$c \leq v_1 \leq 2^{\frac{1}{p}} c \leq v_2 \leq 1$$

and that φ attains its maximum in $[c, 2^{\frac{1}{p}} c]$ at v_1. In that case, after simplification,

$$\max \varphi(v) = (\frac{2}{p} - 1) c v_1 + \frac{c}{v_1}.$$

Let us suppose now that $0 < u < 2^{-\frac{1}{p}}$. In this case, $2^{\frac{1}{p}} c < v \leq 1$, and by Theorem 3.2, we know that

$$\begin{aligned} f'(0) &\leq u(1 - v^2) + vu((\frac{2}{p} + \log \frac{1}{2^{\frac{2}{p}} u^2}) \\ &= \frac{c}{v} - cv + \frac{2c}{p} - 2c \log(\frac{2^{\frac{1}{p}} c}{v}) \\ &= \psi(v). \end{aligned}$$

ψ is easily seen to be decreasing and therefore attains its maximum at $2^{\frac{1}{p}} c$, and

$$\psi(2^{\frac{1}{p}} c) = 2^{-\frac{1}{p}} - c^2 2^{\frac{1}{p}} + \frac{2c}{p}.$$

Comparing this value with $\varphi(v_1)$, we conclude that $\varphi(v_1) \geq \psi(2^{\frac{1}{p}}c)$. Taking

$$\begin{aligned} f(z) &= \frac{v+z}{1+vz}((\frac{c}{v})^{\frac{p}{2}} + \sqrt{1-(\frac{c}{v})^p z})^{\frac{2}{p}} \\ &= (\frac{c}{v})(1+vz)^{\frac{2}{p}-1}(v+z) \end{aligned}$$

gives a function in H^p with norm 1 such that $f(0) = c$ and

$$f'(0) = (\frac{2}{p} - 1)cv + \frac{c}{v},$$

where v is the unique solution $(0 < v \leq 1)$ of $v^p - c^p = c^p v^2$.

When $n > 1$, we use $\tilde{f}$ as described in Section 2 to show that $M_p(n, c) = M_p(1, c)$ and the corresponding extremal function is

$$f(z) = (\frac{c}{v})(1+vz^n)^{\frac{2}{p}-1}(v+z^n)$$

where v is such that $v^p - c^p = c^p v^2$. □

6. Corollaries

Corollary 6.1. *If* $1 \leq p < \infty$, $\|f\|_p = 1$, $|f(0)| < 2^{-\frac{1}{p}}$ *and*

$$|f'(0)| > |f(0)|(\frac{2}{p} + \log\frac{1}{2^{\frac{2}{p}}|f(0)|^2}),$$

then f *has at least one zero in the open unit disk.*

Proof. This result follows directly from Theorem 3.2. □

Corollary 6.2. *Let* $p \geq 1$. *Then*

$$\sup\{|f'(0)| : f \in H^p, ||f||_p \leq 1, f \text{ zero-free } \} = (\frac{2}{e})^{\frac{1}{q}}.$$

Proof. If $f \in H^p$ is non-zero and $\|f\|_p \leq 1$, then let $f(0) = c$ for some $0 < c \leq 1$. By Theorems 3.1 and 3.2,

$$|f'(0)| \leq \begin{cases} c(\frac{2}{p} + \log\frac{1}{2^{\frac{2}{p}}c^2}) & 0 < c < 2^{-\frac{1}{p}} \\ \frac{2}{p}c^{1-\frac{p}{2}}\sqrt{1-c^p} & 2^{-\frac{1}{p}} \leq c \leq 1 \end{cases}$$

By varying c between 0 and 1, it is straightforward to show that the maximum derivative is obtained at $c = \frac{1}{2}(\frac{2}{e})^{\frac{1}{q}}$ and the corresponding extremal function is

$$f(z) = (\frac{(1+z)^2}{2})^{\frac{1}{p}}(\exp\frac{z-1}{z+1})^{\frac{1}{q}}$$

as conjectured by Hummel, Scheinberg, and Zalcman in [5]. □

This result was first obtained by Johnny Brown in [4] by a completely different method.

Corollary 6.3. *Let* $2 < p < \infty$ *and* q *be such that* $\frac{1}{p} + \frac{1}{q} = 1$. *Then there does not exist* $0 < r < 1$ *such that*

$$\|f\|_p = (\int_{-\pi}^{\pi} \frac{1}{2\pi} |f(e^{it})|^p dt)^{\frac{1}{p}} \geq (\sum_{n=0}^{\infty} |\hat{f}(n)|^q r^n)^{\frac{1}{q}}$$

for every $f \in H^p$.

Proof. Consider the extremal function f in Theorem 3.1 for $n = 1$. We have $\|f\|_p = 1$. For $2^{-\frac{1}{p}} \leq c < 1$, $f(0) = c$, and

$$f'(0) = \frac{2}{p} c^{1-\frac{p}{2}} \sqrt{1 - c^p} > A\sqrt{1 - c}$$

if c is close enough to 1, where A is a positive constant. Therefore, since $q < 2$,

$$\sum_{k=0}^{\infty} |\hat{f}(k)|^q r^k \geq c^q + A^q r (1 - c)^{\frac{q}{2}} > 1$$

for c sufficiently close to 1. □

Notice that this corollary implies that no Bohr phenomenon holds for H^p and L^p ($2 < p < \infty$.) This was also proved independently by L. Aizenberg [1] using a different method. It is known that the Bohr phenomenon does hold for real L^∞. For L^∞, the Bohr phenomenon does not hold as shown by an example of Liflyand's [6]. Whether such a Bohr phenomenon holds for real L^p ($2 < p < \infty$) remains an open problem.

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