THE ISOPERIMETRIC INEQUALITY VIA APPROXIMATION THEORY AND FREE BOUNDARY PROBLEMS

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ABSTRACT. In this survey paper, we examine the isoperimetric inequality from an analytic point of view. We use as a point of departure the concept of *analytic content* in approximation theory: this approach reveals ties to overdetermined boundary problems and hydrodynamics. In particular, we look at problems connected to determining the shape of an electrified droplet or equivalently, that of an air bubble in fluid flow. We also discuss the connection with the Schwarz function and quadrature domains. Finally, we survey some known generalizations to higher dimensions and list many open problems that remain. This paper is an expanded version of the plenary talk given by the second author at the fifth CMFT conference in Joensuu, Finland, in June 2005.

1. HISTORY

The isoperimetric theorem deals with the question of finding, among all simple closed curves of a given length P, the curve that surrounds the largest area. It was already known to the Greeks that the answer is the circle of radius $R = P/2\pi$: Pappus and later Theon of Alexandria (III A.D.) already state the result and give credit to Zenodorus (see [4, 6] for a historical discussion). We thus obtain the isoperimetric inequality:

$$Area = A \leq \pi(P/2\pi)^2,$$

or equivalently,

$$4\pi A \leq P^2. \tag{1.1}$$

"Cutting a hole" in a domain will increase P and diminish A, so (1.1) holds for multiply connected domains as well.

J. Steiner [59] in 1838 gave the first proof of the result assuming the solution exists. It was completed by P. Edler [15] in 1882, H.A. Schwarz [55] in 1884 and C. Carathéodory and E. Study [8] in 1910. Schwarz also proved the three dimensional version $S^3 \geq 36\pi V^2$. E. Schmidt [54] in 1938 published a proof of the n-dimensional version:

$$S^n \geq 2\pi^{n/2} n^{n-1} V^{n-1}/\Gamma(n/2).$$

A survey by Bläsjö [6], giving in particular a nice account of the early geometric "proofs" of the isoperimetric theorem, has recently appeared.

2000 *Mathematics Subject Classification.* Primary: 30H05, Secondary: 35R35.

The second author gratefully acknowledges support from the National Science Foundation.

One of the early analytic proofs of (1.1) was given by A. Hurwitz [26] in 1901. Let us sketch his argument here.

Sketch of proof. Suppose the region Ω is bounded by the simple closed smooth curve Γ, parametrized with respect to the arc-length parameter s and with length 2π. (So, the isoperimetric inequality would state that $A \leq \pi$.) That is, we can write

$$\Gamma := \{z(s) = \sum_{-\infty}^{\infty} c_n e^{ins}\},$$

where $|z'(s)| = 1$. Then

$$1 = \frac{1}{2\pi} \int_0^{2\pi} |z'(s)|^2 ds = \sum_{-\infty}^{\infty} n^2 |c_n|^2.$$

On the other hand, using Green's theorem,

$$\begin{aligned} A &= \frac{1}{2} \int_\Gamma (xdy - ydx) \\ &= \frac{1}{2} Im \int_\Gamma \bar{z}(s) z'(s) ds \\ &= \pi \sum_{-\infty}^{\infty} n |c_n|^2. \end{aligned}$$

Since $n \leq n^2$, $A \leq \pi$. Moreover, $A = \pi \Leftrightarrow c_n = 0, n \neq 0, 1$, i.e.

$$\Gamma := \{c_0 + c_1 e^{ins}\}, \text{ a circle.}$$

□

Another proof using the Riemann mapping theorem was given by Carleman in 1921.

Theorem 1.1. *([9])*

Suppose Ω is a Jordan domain with a rectifiable boundary. Then for all f analytic in Ω and continuous in the closure of Ω,

$$\int\int_\Omega |f|^2 dA \leq \frac{1}{4\pi} (\int_\Gamma |f| ds)^2.$$

Notice that letting $f = 1$ yields the isoperimetric inequality. Also note that the above inequality would be rather trivial if we had the expres sion $\frac{1}{4\pi} \int_\Gamma |f|^2 ds$ on the right hand side; that inequality can be obtained immediately from the similar inequality in the unit disk via conformal mapping. The Carleman inequality, however, requires an additional neat trick beyond a change of variables.

In this survey paper, we are interested in the analytic approach to the isoperimetric inequality and its connections with approximation theory and free boundary problems. This approach reveals a close tie to hydrodynamics and, in particular, to problems concerning shapes of electrified droplets of perfectly conducting fluid. We use as a point of departure the paper [32], in which the author discusses the concept of analytic content, and the related survey paper [18]. In Section 3, we discuss the connection with overdetermined boundary value problems and Serrin's theorem. In Section 4, we describe a more general problem and its application to determining the shape of a droplet of conducting fluid in the presence of an electric field. In Section 5, we examine some special cases of this more general problem in connection with the Schwarz function, Vekua's problem, and quadrature domains. Finally, in Section 6, we survey known generalizations to higher dimensions and state many remaining open problems.

Acknowledgements. The authors would like to thank Dr. Van Deursen and Dr. Darren Crowdy for pointing out some interesting references to electrowetting. We are also indebted to the referee for several thoughtful remarks helping to improve the exposition.

2. Analytic Content

Let Ω be a finitely connected region in $\mathbb{C}$ bounded by n simple closed analytic curves γ_j, $j = 1, \ldots, n$. Let $\Gamma = \cup\gamma_j$ be the boundary of the region Ω. A_Ω denotes the set of functions analytic in Ω and continuous in $\bar{\Omega}$, and $C(\bar{\Omega})$ is the set of continuous functions in $\bar{\Omega}$. Let $R(\bar{\Omega})$ denote the closure (in the uniform norm) of the set of rational functions with poles outside of $\bar{\Omega}$. In general, not every function analytic in Ω and continuous in $\bar{\Omega}$ is an element of $R(\bar{\Omega})$, but it is well-known that for the finitely connected regions Ω with analytic boundaries that we are considering here, $A_\Omega = R(\bar{\Omega})$ (see, for example, [16]). In general, it is an interesting and central question of approximation theory to determine when a function is an element of $R(\bar{\Omega})$, or what its distance to $R(\bar{\Omega})$ is. In particular, we will be interested in approximating the "simplest" non-analytic function, namely $\bar{z}$. The following definition was introduced in [31].

Definition. *The analytic content of a domain Ω is*

$$\lambda(\Omega) := \inf_{\varphi \in A_\Omega} \| \bar{z} - \varphi \|_{C(\bar{\Omega})} .$$

Therefore the analytic content of a compact set $\bar{\Omega}$ may be thought of as the distance between the function $\bar{z}$ and the algebra $R(\bar{\Omega})$, or in the case of finitely connected domains, A_Ω. The following theorem gives a quantitative geometric estimate of the analytic content of a set - and also yet another proof of the isoperimetric inequality!

Theorem 2.1. *Let Ω and Γ be as above, and let A and P be the area and perimeter of Ω, respectively. Then*

$$\frac{2A}{P} \le \lambda(\Omega) \le \sqrt{\frac{A}{\pi}},$$

so $P^2 \ge 4\pi A$. Moreover, $\lambda(\Omega) = \sqrt{\frac{A}{\pi}} \Leftrightarrow \Omega$ is a disk.

This theorem is discussed in detail in [18]. Let us outline the argument here. H. Alexander in 1973 ([3]) proved the upper estimate by noticing the connection with the Ahlfors-Beurling estimate from 1950 ([1]).

More specifically, suppose D is a bounded domain (with smooth boundary ∂D) containing $\bar{\Omega}$. By the Cauchy-Green formula,

$$\bar{\zeta} = \frac{1}{2\pi i}\int_{\partial D}\frac{\bar{z}}{z-\zeta}dz - \frac{1}{\pi}\int_D \frac{1}{z-\zeta}dA(z),$$

where dA is area measure. Define

$$G(\zeta) = \frac{1}{\pi}\int_{\bar{\Omega}}\frac{1}{z-\zeta}dA(z).$$

Then

$$\bar{\zeta} + G(\zeta) = \frac{1}{2\pi i}\int_{\partial D}\frac{\bar{z}}{z-\zeta}dz - \frac{1}{\pi}\int_{D-\bar{\Omega}}\frac{1}{z-\zeta}dA(z).$$

The right hand side can be easily seen to be in A_Ω. Therefore

$$\lambda \leq \max_{\zeta\in\bar{\Omega}}|G(\zeta)|.$$

The Ahlfors-Beurling estimate shows that

$$|G(\zeta)| \leq \sqrt{\frac{A}{\pi}},$$

with equality holding only for a disk, thus proving the theorem. Gamelin and D. Khavinson gave a simple proof of the Ahlfors-Beurling estimate (see [18, p. 25-28]). Their argument goes as follows. Without loss of generality, $\max_{\zeta\in\bar{\Omega}}|G(\zeta)|$ is attained at $z=0$, and, rotating, equals

$$\frac{1}{\pi}\mathrm{Re}\int_\Omega \frac{dA(\zeta)}{\zeta} = \lambda_{max}.$$

Now notice that $\{\zeta : \mathrm{Re}\frac{1}{\zeta} \geq c\} = \{\zeta : |\zeta - \frac{1}{2c}| \leq \frac{1}{2c}\} =: \Delta$, a disk. For this disk, $\lambda = \frac{1}{2c} = \sqrt{\frac{A(\Delta)}{\pi}}$, while it is easy to show that, for any $\Omega \neq \Delta$, the integral is smaller, as desired.

D. Khavinson proved the lower estimate in [31] by applying the complex form of Green's theorem to the function $\bar{z}-\varphi(z)$, where $\varphi \in A_\Omega$ is the best approximation to $\bar{z}$. This gives

$$\int_\Gamma (\bar{z}-\varphi(z))dz = 2i\int_\Omega dA(z) = 2iA,$$

so that

$$\lambda P \geq |\int_\Gamma (\bar{z}-\varphi(z))dz| \geq 2A.$$

Theorem 2.1 leads naturally to the following:

Question. *For which Ω does the equality $\frac{2A}{P} = \lambda(\Omega)$ hold?*

The next theorem and following remarks are taken from [32].

Theorem 2.2. *([32]) Let Ω and Γ be as above. The following are equivalent:*
(i) $\lambda = \frac{2A}{P}$;
(ii) There is $\varphi \in A_\Omega$ such that $\bar{z}(s) - i\lambda\bar{\dot{z}}(s) = \varphi(z(s))$ on Γ, where s is the arc-length parameter;
(iii) $\frac{1}{A}\int_\Omega f dA = \frac{1}{P}\int_\Gamma f ds$ for all $f \in A_\Omega$.

Remark. Note that (iii) holds for annuli $\Omega = \{r < |z| < R\}$. Simply take the Laurent series decomposition of $f = f_1 + f_2$ in the annulus, where f_1 is analytic inside $\{z : |z| < R\}$ and $f_2(\infty) = 0$, and notice that both sides of the equality in (iii) are equal to $f_1(0)$.

Also, (ii) easily implies that if Γ contains a circular arc, Ω is a disk or an annulus. Indeed, suppose, for simplicity, that Γ contains an arc centered at the origin and of radius R. Then we can write $z(s) = Re^{is/R}$, where s is the arc-length parameter, and a calculation shows that

$$\bar{z}(s) - i\lambda\bar{\dot{z}}(s) = \frac{R^2 - R\lambda}{z(s)} = \frac{c}{z} = \varphi(z)$$

on that arc. But since $\varphi \in A_\Omega$, this equality must hold on all of Γ. If $c = 0$, then $\varphi = 0$ and $|z| = const$ on Γ, so Γ is a disk centered at the origin. If $c \neq 0$,

$$i\lambda z\bar{\dot{z}}(s) = |z|^2 - c,$$

or, in other words, $\mathrm{Re}(z\bar{\dot{z}}(s)) = \frac{1}{2}\frac{d|z|^2}{ds} = 0$. Therefore $|z|^2$ is a constant locally on $\partial\Omega$, so Ω is an annulus.

The following conjecture remains open.

Conjecture 2.1. *([32])*

$$\lambda = \frac{2A}{P} \Leftrightarrow \Omega \textit{ is a disk, or an annulus.}$$

In the simply-connected case, we know more.

Theorem 2.3. *([32])*

$$\textit{If } \lambda = \frac{2A}{P} \textit{ and } \Omega \textit{ is simply connected, } \Omega \textit{ is a disk.}$$

Sketch of proof. Since $\lambda = \frac{2A}{P}$, there exists $\varphi \in A_\Omega$ such that

$$\bar{z}(s) - i\lambda\bar{z}'(s) = \varphi(z(s)). \tag{2.1}$$

Differentiating, we obtain

$$\bar{\dot{z}}(1 + \overline{\frac{i\lambda\ddot{z}}{\dot{z}}}) = \varphi'(z)\dot{z}. \tag{2.2}$$

Since $\dot{z}, \ddot{z}$ are orthogonal vectors, we can rewrite (2.2) as

$$\mathrm{Re}(1 + \frac{i\lambda\ddot{z}}{\dot{z}}) = \varphi'(z)(\dot{z})^2. \tag{2.3}$$

The left hand side of equation (2.3) is real and therefore has argument increment 0, while the right hand side has an argument increment of at least 4π as we travel along Γ, unless $\varphi' = 0$. (Note that we need to be careful if the expression on the left hand side passes through a zero on $\partial\Omega$.) Hence, $\varphi = const$ and (2.1) implies that $\Gamma = \{z : |z - const| = \lambda\}$, a disk. □

Equation (2.1) is closely connected to the "Riccati equation": since Γ is analytic, near each component γ_j there is a single-valued branch of an analytic function $S(z)$ (the Schwarz function, see [2, 12, 57] and also Section 5) such that $\bar{z} = S(z)$ on Γ. Then

$$u := \sqrt{S'(z)}$$

is a single-valued analytic function in a tubular neighborhood of $\partial\Omega$, and

$$u(z) = \frac{d\bar{z}}{ds} \textit{ on } \Gamma,$$

so, after differentiating one more time with respect to the arc-length parameter, (2.1) becomes the Riccati equation

$$u^2 - i\lambda u' = f, \tag{2.4}$$

where $f = \varphi'$. Since Riccati's equation is easily transformed into a homogeneous second order linear equation (see [27]) which may only have two linearly independent solutions, it is yet another indication that if (2.1) holds on Γ, Ω must be at most doubly connected and disks and annuli are the only domains for which (2.1) may hold. Yet Conjecture 2.1 is still open even for doubly connected domains! The Riccati equation (2.4) appears in many free boundary problems, some of which we will discuss in more detail in Section 4. We now turn to the connection with overdetermined boundary problems and Serrin's theorem.

3. Overdetermined Boundary Value Problems and Serrin's Theorem

Recall that condition (*iii*) from Theorem 2.2 states:

$$\frac{1}{A}\int_\Omega f dA = \frac{1}{P}\int_\Gamma f ds$$

for analytic f in Ω. If Ω is simply connected, this condition is equivalent to:

$$\frac{1}{A}\int_\Omega u dA = \frac{1}{P}\int_\Gamma u ds \tag{3.1}$$

for all functions u harmonic in Ω. Moreover, A. Kosmodem'yansky showed that condition (3.1) is equivalent to the following.

Theorem 3.1. *([39]) Consider the solution v of the Dirichlet problem*

$$\Delta v = 1 \textit{ in } \Omega; v = 0 \textit{ on } \Gamma.$$

Then the normal derivative of v must satisfy $v_n = A/P$ on Γ.

Indeed, take any harmonic test function u in Ω that is smooth up to the boundary. By Green's formula,

$$\int_\Gamma u v_n ds = \int_\Omega u dA = \frac{A}{P}\int_\Gamma u ds.$$

Since u is arbitrary, $v_n = A/P$ on Γ. In this context, the shape of Ω was already known. We state the following result due to Serrin in two dimensions, although the theorem is more general and holds in all dimensions.

Theorem 3.2. ([56]) *If the overdetermined boundary value problem*

$$\Delta v = 1 \textit{ in } \Omega,$$

$$v = 0 \textit{ on } \Gamma,$$

$$v_n = const \textit{ on } \Gamma,$$

has a smooth solution in Ω, then Ω is a disk.

This leads to an equivalent form of Conjecture 2.1 "à la Serrin" ([35]):

Conjecture 3.1. *Let Ω be a multiply connected domain. If the overdetermined boundary value problem ($n \geq 2$)*

$$\Delta v = 1 \textit{ in } \Omega$$

$$\frac{\partial v}{\partial n} = \frac{A}{P} \textit{ on } \Gamma$$

$$v|_{\gamma_j} = c_j, j = 1, \ldots, n, c_n = 0$$

has a smooth solution in Ω, then Ω must be an annulus.

Serrin's theorem has a natural interpretation in terms of hydrodynamics (see [36, p. 2-3] and [41, p. 653-654]). Suppose we have a viscous, incompressible Newtonian fluid flowing through a pipe with cross-section Ω. A Newtonian fluid is a fluid in which the shear stress, that is, the stress exerted by the fluid that is tangential to the wall of the pipe, is proportional to the velocity gradient in the direction perpendicular to the plane of shear. (We will see shortly in the discussion below why this is true.) The constant of proportionality is known as the *viscosity* μ. Intuitively, the viscosity measures how "sticky" the fluid is. A viscous fluid is therefore one such that $\mu \neq 0$, like oil or tar. A fluid is incompressible means the density of the fluid is constant along its flow lines; because of the continuity equations in fluid dynamics that express conservation of mass, this implies that the divergence of the velocity vector $\vec{v}$ is zero, i.e., $\mathrm{Div}(\vec{v}) = \nabla \cdot \vec{v} = 0$. Now let us assume in addition that the fluid is flowing in lines parallel say to the $\vec{k}$ axis (such a flow is called *laminary*) and that the flow is steady, i.e., $\frac{d\vec{v}}{dt} = 0$. Then the continuity equation implies that $\vec{v} = v(x, y)\vec{k}$ depends only on x and y. Then the Navier-Stokes equations in this simplified context can be written as

$$\Delta v = -\frac{1}{\mu}\frac{\partial p}{\partial z},$$

where p is the pressure at each point (x, y, z) and depends only on z, since the flow is laminary. Since $\frac{\partial^2 p}{dz^2} = 0$, the right hand side is actually a constant A, giving us (modulo a constant multiple) the first equation in Serrin's theorem.

On the other hand, the force exerted by the water on the walls of the pipe is given by

$$\vec{F} = (p - \frac{4}{3}\,\mu\,\nabla \cdot \vec{v})\,\vec{n} + \mu(\vec{n} \times (\nabla \times \vec{v})),$$

where $\vec{n}$ is an outward unit normal vector (see [36, 41]). This breaks the force $\vec{F}$ down into a normal component and a tangential, or shear, component. Since $\vec{v} = v\vec{k}$ only has a non-zero contribution v in the $\vec{k}$ direction and $\nabla \cdot \vec{v} = 0$, the force $\vec{F}$ simplifies to:

$$\vec{F} = p\,\vec{n} + \mu\frac{\partial v}{\partial n}\,\vec{k}.$$

In other words, the quantity $\mu\frac{\partial v}{\partial n}$ represents the "shear stress" on the wall, which is to be expected, as discussed earlier, for a Newtonian fluid. Finally, because we are dealing with a viscous fluid, and the walls of the pipe are fixed (not moving), the condition that $v = 0$ on Γ is the so-called "adherence" condition.

Therefore Serrin's theorem says that the shear stress at each point on the wall is the same if and only if the cross-section of the pipe is a disk.

H. Weinberger ([61]) gave an alternative proof of Serrin's theorem based on the strong maximum principle for the auxiliary function

$$\varphi := |\Delta v|^2 + v$$

and a Rellich type identity. Serrin and Weinberger's methods were extended to more general equations and boundary conditions but always with the additional provision that the solution of the overdetermined boundary value problem in question does not achieve local extrema inside the domain. For example, the following result combines the efforts of J. Serrin ('71), G. Alessandrini ('92), W. Reichel ('96), N. Willms, M. Gladwell, and D. Siegel ('94), B. Sirakov ('01), L. Payne and P. Schaeffer ('89), and also G. Philippin and L. Ragoub ('95) (see [42, 43, 44, 45, 46, 47, 48] and references therein).

Theorem 3.3. *The overdetermined boundary value problem*

$$\begin{aligned} \Delta v &= 1 \\ \frac{\partial v}{\partial n} &= a_j \\ v|_{\gamma_j} &= c_j, j = 1, \ldots, n, \end{aligned}$$

$c_n = 0, a_n \geq 0, c_j < 0, a_j \leq 0$ *has a solution if and only if* Ω *is a disk or an annulus, and, accordingly,* v *is a radial function.*

However, our isoperimetric problem equivalent to the overdetermined boundary value problem (Conjecture 3.1) has all $a_j = \frac{A}{P}$, hence v certainly attains its minimum inside Ω. Therefore the methods developed to prove results like Theorem 3.3 do not work.

We now turn to a discussion of condition (ii) in Theorem 2.2 and a related application to determining the shape of droplets of conducting fluid in the presence of an electric field.

4. Droplets

Recall one of the equivalent conditions for $\lambda(\Omega) = 2A/P$:

$$\bar{z}(s) - i\lambda\dot{\bar{z}}(s) = \varphi(z(s))$$

for some $\varphi \in A_\Omega$. We would like to consider a more general problem, in which the function φ may not be continuous in the closure of Ω and may possibly have poles inside Ω. However, we will still need to consider functions that are relatively "good" near the boundary. In order for polynomials or rational functions to be dense in the appropriate space of analytic functions, we will also need to require some a priori regularity of the domain itself. Let us begin therefore with a discussion of Smirnov classes and Smirnov domains.

Smirnov classes are one of the two standard generalizations of Hardy classes to arbitrary domains. Let us suppose Ω is a finitely connected domain in $\mathbb{C}$ bounded by rectifiable Jordan curves. We say a function f analytic in Ω belongs to the Smirnov class $E^1 := E^1(\Omega)$ if there exists a constant M and an increasing sequence of domains Ω_k with boundaries Γ_k that consist of a finite number of rectifiable Jordan curves such that

$$\cup_k \Omega_k = \Omega,$$

and

$$\sup_k \int_{\Gamma_k} |f(z)| \, ds \leq M < \infty.$$

In this definition, the domains and their boundaries appear to depend on the function, and therefore it is not obvious that E^1 is even a linear space. However a theorem of Keldysh and Lavrentiev ([29]) for simply connected domains, extended by S. Ya. Khavinson and Tumarkin ([38]) to finitely connected domains, ensures that this dependence is superfluous. It follows that the function f has non-tangential boundary values $f^*(\zeta)$ almost everywhere on Γ, with $f^* \in L^1(\Gamma)$. In addition, the Cauchy integral formula holds for f:

$$f(z) = \frac{1}{2\pi i} \int_\Gamma \frac{f^*(\zeta)}{\zeta - z} d\zeta$$

for all $z \in \Omega$. In fact, the existence of an analytic function's representation as a Cauchy integral of its non-tangential boundary values can be taken as an equivalent definition of its membership in E^1, in domains with rectifiable boundaries. Defining $\|f\|_{E^1} = \|f^*\|_{(L^1(\Gamma),ds)}$ makes E^1 equipped with this norm into a Banach space. For a more complete discussion of Smirnov classes and generalized Hardy spaces, see [13, Chapter 10], [20, Chapter X]. See also the Appendix in [37, p. 57-61].

Now let us consider a conformal mapping φ of Ω onto a domain whose boundary consists of finitely many circles or points. Such a conformal mapping always exists by Koebe's theorem (see [20, p. 237-238]). Since Ω has a rectifiable boundary Γ, φ' can be shown to be in $E^1(\Omega)$. We say that Ω is a *Smirnov* domain if, for each $z \in \Omega$,

$$\log|\varphi'(z)| = \frac{1}{2\pi}\int_\Gamma \log|\varphi'(\zeta)|\frac{\partial g_\Omega(\zeta, z)}{\partial n}|d\zeta|,$$

where $g_\Omega(\zeta, z)$ is the Green function of Ω having singularity at z. In other words, φ' has no singular part and the harmonic function $\log|\varphi'|$ can be recovered from its boundary values via the Poisson formula. Let us now state a more general problem.

Problem 4.1. *Find all Smirnov domains $\Omega \in \hat{\mathbb{C}} := \mathbb{C} \cup \infty$, whose boundary $\Gamma = \bigcup_1^n \gamma_j$, consists of n rectifiable Jordan curves, such that there exists $F(z) \in E^1(\Omega)$ analytic, or F meromorphic in Ω and in E^1 close to the boundary of Ω and with prescribed poles $z_1, \ldots, z_k$ in Ω, such that*

$$F(z) = p_j\bar{z} + i\tau_j\dot{\bar{z}} + c_j \text{ on } \gamma_j,$$

for some constants $p_j, \tau_j \in \mathbb{R}, p_j^2 + \tau_j^2 \neq 0$, $c_j \in \mathbb{C}$. (If $\infty \in \Omega$ we will always assume it to be one of the poles.)

This problem turns out to be interesting in determining the shape of electrified droplets of fluid: suppose Γ is the boundary of a planar droplet of perfectly conducting fluid in the presence of an electrostatic field E. Let p be the fluid pressure inside the droplet and τ the surface tension, and call Ω the unbounded component of the droplet. E has a harmonic potential u, that is, $E = -\nabla u$, and u is the real part of an *analytic* potential g. This analytic potential, after various normalizations (see [36, p. 15]), can be written as

$$g(z) = z + \frac{\alpha_1}{z} + \frac{\alpha_2}{z^2} + \ldots$$

P. Garabedian ([17]) showed that if the droplet is in equilibrium,

$$F(z) = p\bar{z} + i\tau\frac{d\bar{z}}{ds} \text{ on } \Gamma, \tag{4.1}$$

where

$$F(z) = \int (g')^2 dz = z + \frac{2\alpha_1}{z} + \ldots \tag{4.2}$$

for $z \in \Omega$. F is called an *integrated analytic potential.* In [36], every solution to (4.1) on its Jordan boundary Γ that has the form (4.2) for $z \in \Omega$ is defined to be a *mathematical droplet*. For a *"physical droplet"*, $g' = \sqrt{F'}$ must be single valued in Ω. If τ is very large, the p-term in (4.1) is negligible, and the resulting equation also describes small air bubbles in fluid flow ([17, 40]). A lower bound for τ should exist as well. For $\tau = 0$ there are certainly no physical droplets. Yet, mathematical droplets for that case are ellipses ([12, 36, 57]). The following result shows the existence of a one parameter family of mathematical droplets.

Theorem 4.2. *(*[36, p. 24-26]*) There exists a one parameter family of unbounded domains Ω_t, each with rectifiable boundary Γ_t, and a corresponding family of functions F_t analytic in Ω_t except for a simple pole with residue 1 at ∞, such that*

$$F_t(z) = p_t\bar{z} + i\tau_t\frac{d\bar{z}}{ds} \text{ on } \Gamma_t,$$

for some real constants p_t and τ_t, with $p_t \neq 0$, $\tau_t \neq 0$.

Each of these domains Ω_t is thus an example of a solution to Problem 4.1. Their boundaries Γ_t are images of the unit circle under a rational mapping of degree 3 on which (4.1) holds. None of these curves however is a physical droplet. To our knowledge, no other examples of such domains are known. In particular, we do not know of any examples of transcendental curves satisfying (4.1), although, most likely, there are plenty of them!

Applying electrical forces to droplets of conducting fluid has led to some very concrete applications: the process of "electrowetting", for example, in which an electric force is applied at the interface of a droplet of conducting fluid and a solid, has applications to digital cameras, camera phones, and home security systems. In 2003, scientists from Philips Research created a fluid lens that operates on the basis of the process of electrowetting: two non-mixing fluids, one conducting and one not, are placed inside a tube. The layer between the liquids (the *meniscus*) acts as a lens. An electric field is applied to the tube, which causes the conducting fluid to change its shape, thus resulting in a change of the focal length of the lens. See [49] for more details. For further references on electrowetting and its applications, see [5, 24]. A slightly different type of application can be found in [11]: there, the authors use Schwarz functions to model the changing shape of a void created and traveling inside a thin metal conductor subjected to an intense electric field. This model is similar in some ways to the one used for Hele-Shaw flows (see [10, 23, 53]).

5. Some special cases

Let us now examine three distinguished cases of Problem 4.1, in which the boundary condition on $\Gamma = \cup_{j=1}^{n}\gamma_j$ simplifies to one of the following:

$$F(z) = p_j\bar{z} \qquad z \in \gamma_j,\ p_j \in \mathbb{R} - \{0\}; \tag{5.1}$$

$$F(z) = i\tau_j\dot{\bar{z}} \qquad z \in \gamma_j,\ \tau_j \in \mathbb{R} - \{0\}; \tag{5.2}$$

$$F(z) = p_j\bar{z} + c_j \qquad z \in \gamma_j,\ p_j \in \mathbb{R} - \{0\}, c_j \in \mathbb{C}. \tag{5.3}$$

Note that the existence of a function F satisfying (5.2) implies the existence of a function g satisfying (5.3): simply define

$$g(z) = \int (F(z))^2 dz.$$

Then, by (5.2), for $z \in \gamma_j$, we have

$$\int (F(z))^2 dz = -\tau_j^2 \int (\frac{d\bar{z}}{ds})^2 dz = -\tau_j^2(\bar{z} + c_j),$$

for some constant c_j. Therefore g is well-defined as a single valued analytic function, and (5.3) holds. From now on, we shall always assume additional regularity for Ω, i.e., that Ω is a Jordan Smirnov domain.

5.1. **The Schwarz function.** Notice that the first special case (5.1) is intimately connected with the Schwarz function, since on each boundary component, F/p_j is the Schwarz function for γ_j. More specifically, let us recall the definition of the Schwarz function of a curve (see [12, 57]):

Definition. *Suppose Γ is a non-singular real-analytic Jordan arc in $\mathbb{C}$. Then there is a neighborhood G of Γ and a uniquely determined analytic function S on G such that*

$$S(z) = \bar{z} \quad \textit{for } z \in \Gamma.$$

S is called the Schwarz function of Γ.

(5.1) then means that all of the Schwarz functions F/p_j of the curves γ_j are connected to each other, each one being a real multiple of F. The following simple theorem is stated in [36]:

Theorem 5.1. *(*[36, Thm 5.3]*) If Ω is bounded and* (5.1) *holds with $F \in E^1(\Omega)$, then Ω must be an annulus.*

Sketch of proof. Since $f \in E^1(\Omega)$ and Ω is bounded, $zF(z) \in E^1(\Omega)$. Because of the boundary condition (5.1), $zF(z) = p_j|z|^2$ on γ_j, so $zF(z)$ is also in $L^\infty(\Gamma)$. Finally, since Ω is a Smirnov domain, the above two conditions imply that $zF(z) \in H^\infty(\Omega)$ (see [13]). But $zF(z)$ is analytic in Ω and real-valued on Γ, and therefore $zF(z)$ is constant. (Notice that the condition that Ω is Smirnov is crucial here.) Therefore $|z|$ is constant on each boundary component, so Ω must be an annulus or a disk centered at the origin. Because $F(z) = c/z$ is not analytic in the disk, Ω must be an annulus. □

Note that the coefficients p_1 and p_2 in the above theorem are not arbitrary: they are connected to the radii of the annulus, and are not equal to each other. If we require the coefficients p_j to be equal, therefore, there is no analytic function that satisfies (5.1) for a bounded domain Ω. On the other hand, if we allow F to have poles $a_1, \dots, a_m$, and require the coefficients p_j to be equal (without loss of generality $p_j = 1$), then Ω is a so-called *quadrature domain*; namely, if f is any function analytic in Ω, then by the complex form of Green's theorem,

$$\int_\Omega f dA = \frac{1}{2i}\int_\Gamma f\bar{z}dz = \frac{1}{2i}\int_\Gamma fFdz = \pi\sum_{j=1}^m f(a_j)\mathrm{Res}_{a_j}F.$$

These domains have been intensely studied in the 1980s by D. Aharonov, B. Gustafsson, H. S. Shapiro, K. Ullemar, Y. Avsi (see [57] and references therein). Also, see [23] for an account of many recent developments.

Even when we do not require the coefficients p_j to be equal, a similar argument as in the proof of Theorem 5.1 shows that if F is assumed to have a simple pole at the origin and Ω is bounded, then Ω must be a disk. (This is well-known in the context of Schwarz functions: the Schwarz function of a domain has one pole if and only if the domain is a disk.) If the function F has two different poles (and if the coefficients p_j are different), then the problem is already more difficult.

5.2. **Vekua's Problem.** The second special case (5.2)

$$F(z) = i\tau_j\bar{\dot{z}} \qquad z\in\gamma_j,\ \tau_j\in\mathbb{R}-\{0\}$$

is a particular example of an overdetermined boundary value problem made enormously popular by works of I. N. Vekua in the 1950s. It is not difficult to see that (5.2) implies (see [60]) that F is orthogonal on Γ to all functions analytic in Ω with a single-valued primitive, hence

$$F(z) = \sum_{j=1}^{n-1} c_j\frac{\partial\omega_j}{\partial z},$$

for some real constants c_j, where ω_j are the harmonic measures of the boundary components γ_j with respect to Ω, i.e., ω_j ($j = 1,\dots,n-1$) are harmonic inside Ω and equal to 1 on γ_j and 0 on $\Gamma-\gamma_j$. Define

$$u(z) = \sum_{j=1}^{n-1} c_j\omega_j(z).$$

Then u solves the following overdetermined boundary value problem in Ω:

$$(V)\quad\begin{cases}\Delta u = 0\\ u = c_j\ on\ \gamma_j\,(j=1,\dots,n-1),\ c_n = 0\\ \frac{\partial u}{\partial n} = \tau_j\ on\ \gamma_j\,(j=1,\dots,n).\end{cases}$$

For this special case of Vekua's overdetermined boundary value problem, Ω is known, in the doubly-connected case:

Theorem 5.2. *(*[36, Thm 5.5]*) If Ω is a doubly-connected bounded Smirnov domain, and (V) is solvable, then Ω is an annulus.*

The problem is open for domains of connectivity 3 and higher. The authors of [36] conjecture that there are no solutions in that case.

Again, if we assume that all the constants on all boundary components are the same in (5.2), that is,

$$F(z) = i\tau\bar{\dot{z}}, \ \ \tau \neq 0, \ \ z \in \Gamma,$$

and we allow F to have poles $a_1, \ldots, a_m$, then for all f analytic in Ω, we have:

$$\int_\Gamma f\,ds = \frac{1}{i\tau}\int_\Gamma Ff\,dz = \frac{2\pi}{\tau}\sum_1^m f(a_j)\mathrm{Res}_{a_j}F.$$

This type of domain Ω is called an *arc-length quadrature domain* and much is known about them (see [23, 57]). For example, simply connected arc-length quadrature domains are conformal images of the unit disk under maps by rational functions of a rather special form (see [57, Thm 5.4]). Note that the assumption that Ω is Smirnov is crucial, otherwise there exist, for example, one point arc-length quadrature domains, so-called pseudo-circles, with highly non-smooth boundaries (see [29, 51, 58]).

There are many applications of quadrature domains to fluid dynamics. One of the most well-known examples is the connection between quadrature domains and Hele-Shaw flows, which was discovered by S. Richardson in [53]. A detailed survey of applications of quadrature domains to fluid dynamics can be found in [10].

5.3. **Unbounded domains.** Let us mention briefly the case of unbounded (Smirnov) domains Ω, with Jordan boundary Γ. Notice that we are now considering the boundary Γ to consist of a single curve. Recently, P. Jones and S. Smirnov ([28]) showed that if a domain is Smirnov, then its complement is most likely non-Smirnov! This provided an unexpected negative solution to a problem posed by S. Ya. Khavinson in the 50's, which proposed to find an intrinsic characterization of a Smirnov curve: the theorem of Jones and Smirnov revealed therefore that such a characterization is not possible.

If there exists a function F satisfying (5.2), the shape of Γ will depend on the behavior of F at infinity. Let us begin by considering functions analytic at ∞ in Ω.

Theorem 5.3. *([14]) Let Γ be a Jordan curve whose exterior Ω is a Smirnov domain. Suppose there exists $F \in E^1(\Omega)$, analytic at ∞, such that*

$$F(z) = \frac{d\bar{z}}{ds} \ \ a.e. \text{ on } \Gamma. \tag{5.4}$$

Then Γ is a circle and $F(z) = \frac{c}{z}$.

In [14], the authors assumed that $F(\infty) = 0$; in [36], this hypothesis was shown to be redundant. In potential-theoretic terms, this statement is equivalent to saying that if the equilibrium mass distribution is uniformly distributed on Γ (with respect to arc-length), then Γ must be a circle (see [14] for details). This holds in all dimensions provided that Γ is a C^2 surface (see [52]). Equivalently, if the overdetermined boundary value problem

$$\Delta u = 0 \textit{ in } \Omega;$$
$$u = const \neq 0 \textit{ on } \Gamma;$$
$$\frac{\partial u}{\partial n} = const \textit{ on } \Gamma$$

has a solution in an (unbounded!) domain Ω, then Γ is a circle.

Remark. In [14], the authors notice that it is possible to drop the assumption that the domain is Smirnov, but then instead one must assume that the function F is in E^2, since the proof uses the fact that the function $z^2(F(\varphi(z)))^2\varphi'(z)$ is in $H^1(\mathbb{D})$ (where φ is the Riemann mapping from the disk to Ω), and therefore cannot coincide with the conjugate of an H^1 function on the circle. It is not clear whether the theorem itself fails if one drops the assumption that Ω is Smirnov and considers only $F \in E^1$. In this context, one must cautiously observe that in non-Smirnov domains, there exist functions with positive and bounded boundary values which belong to any E^p class, $p < \infty$ (see [30]).

We may also consider the case where F has a simple pole at infinity. Recall that this context has a physical interpretation, discussed in Section 4, as a droplet of conducting fluid in which the surface tension is much larger than the pressure inside the droplet (which is then considered negligible). In this case, the following theorem gives an example of a family of mathematical droplets.

Theorem 5.4. *(*[36, Thm 6.2]*) Let Γ be a Jordan curve, with (logarithmic) capacity* 1*, whose exterior Ω is a Smirnov domain. If $\tau \geq \frac{3+2\sqrt{3}}{3}$ and there exists $F \in E^1$ near the boundary of Ω and with a simple pole at ∞, that is, $F = z + O(\frac{1}{z})$, and*

$$F = i\tau \frac{d\bar{z}}{ds} \textit{ on } \Gamma, \tag{5.5}$$

then Γ is included into one parameter family $\{\Gamma_t\}$, $t = 1/\tau$, where Γ_t is the image of the unit circle under the conformal mapping

$$\varphi_t(w) = \frac{1}{w} - 2tw - \frac{t^2}{3}w^3.$$

For $\tau \leq \frac{3+2\sqrt{3}}{3}$, (5.5) *has no solution among mathematical droplets with Jordan boundaries. The droplets are convex for $\tau \geq 3$ and the family contains only one physical droplet corresponding to the value $\tau = 3$.*

6. Extensions to higher dimensions

Finally, let us discuss what is known in higher dimensions. Suppose Ω is a bounded domain in $\mathbb{R}^n$, $n \geq 3$, Γ is the boundary of Ω, V is the volume of Ω, and P is the (surface) area of Γ. Let $H(\Omega)$ be the closure in the uniform norm on $\bar{\Omega}$ of the space of functions harmonic in a neighborhood of Ω. More generally, if K is a compact subset of $\mathbb{R}^n$, and $C(K)$ is the space of continuous functions on

K, we will write $H(K)$ for the uniform closure in $C(K)$ of the space of functions harmonic in a neighborhood of K.

Let $\mathbf{x} = (x_1, \ldots, x_n)$ be a vector in $\mathbb{R}^n$, and $|\mathbf{x}|^2 = (\sum_{j=1}^n x_j^2)^{\frac{1}{2}}$. If one thinks of $H(K)$ as the uniform closure of the kernel of the Laplace operator Δ and $R(K)$ as the uniform closure of the kernel of the operator $\partial/\partial\bar{z}$, then the analogy of the anti-analytic function $\bar{z}$ is the function $|\mathbf{x}|^2$, since $(\partial/\partial\bar{z})(\bar{z}) = 1$ and $\Delta(|\mathbf{x}|^2) = 2n = const \neq 0$. With this in mind, we define the concept of *harmonic content* as follows.

Definition. *The harmonic content of K is defined to be*

$$\Lambda(K) := dist_{C(K)}(|\,\mathbf{x}|^2, H(K)).$$

For a bounded domain Ω, we will write $\Lambda(\Omega) := \Lambda(\bar{\Omega})$. We then have the following result.

Theorem 6.1. *(*[33]*)*

$$\Lambda(K) = 0 \Leftrightarrow H(K) = C(K).$$

Note that in the case of analytic content in $\mathbb{C}$, the equivalence of the statements $\lambda(K) = 0$ and $R(K) = C(K)$ follows at once from the Stone-Weierstrass theorem, since $R(K)$ is an algebra. However, Theorem 6.1 is non-trivial, since $H(K)$ is not an algebra. Different proofs were given by Poletsky ([50]) and Bliedtner (see [7] and references therein, in particular to the works of W. Hansen).

Harmonic content can be estimated in terms of geometric quantities. If R_{harm} is the radius of the ball with the same capacity as $\bar{\Omega}$, and R_{vol} is the radius of the ball with the same volume as Ω, then the following theorem gives upper and lower bounds for the harmonic content of a domain Ω.

Theorem 6.2. *(*[33, 34]*)*

$$\frac{1}{2}R_{harm}^2 \leq \Lambda(\Omega) \leq \frac{1}{2}R_{vol}^2$$

and equality on either side occurs only for balls.

The upper estimate was proved in [33], and the lower estimate as well as extensions of both inequalities to general elliptic operators were obtained in [34]. An interesting extension of this result to approximation in C^1-norm by harmonic functions is due to Gauthier and Paramonov (see [19]).

Note that the harmonic content $\Lambda(\Omega)$ leads to a different isoperimetric inequality, $R_{harm} \leq R_{vol}$. A lower bound depending only on volume and perimeter cannot occur, because one can construct a "Swiss cheese" set $K \subset \mathbb{R}^2$ that has positive area and finite perimeter, yet so that $H(K) = C(K)$, that is, such that $\Lambda(K) = 0$ (see [25]). In order to get a better hold of the simple geometric quantities of a domain, let us now consider another analogue of the concept of analytic content in higher dimensions.

Recall that analytic content for a domain Ω in $\mathbb{C}$ is defined as

$$\lambda(\Omega) := \inf_{\varphi\in A_\Omega} \| \bar{z} - \varphi \|_{C(\bar{\Omega})} .$$

Note that this is also equal to

$$\lambda(\Omega) := \inf_{\varphi\in A_\Omega} \| z - \bar{\varphi} \|_{C(\bar{\Omega})} .$$

An anti-analytic function $\bar{\varphi} = f_1 + if_2$ can be identified with the harmonic vector field $f = (f_1, f_2) = \nabla u$, u a harmonic real-valued function, where

$$\mathrm{Div}\overrightarrow{f} = \mathrm{Curl}\overrightarrow{f} = 0.$$

(See the discussion in [22, p. 76].) This motivates our definition of analytic content in higher dimensions: define the space $A(\Omega)$ of harmonic vector fields $\overrightarrow{f} = (f_1, \ldots, f_d)$ in Ω as the set of all vector fields $\overrightarrow{f} \in C^1(\Omega) \cap C(\bar{\Omega})$ such that

$$\mathrm{Div}\overrightarrow{f} = \mathrm{Curl}\overrightarrow{f} = 0.$$

We define

$$B(\Omega) = \{\nabla \overrightarrow{h},\, h \in H(\bar{\Omega})\}.$$

Unless Ω is simply-connected,

$$B(\Omega) \subsetneq A(\Omega).$$

Also $\| \overrightarrow{f} \|_\infty = \sup_{x\in\Omega}(\sum_1^n f_j^2(x))^{1/2}$. Then we define the analytic content $\lambda(\Omega)$ of a domain in $\mathbb{R}^n$ as follows.

Definition.

$$\lambda(\Omega) = dist(\overrightarrow{x}, A(\Omega)) := \inf_{f\in A(\Omega)} \|\overrightarrow{x} - f\|_\infty.$$

The analogue of Theorem 2.1 is then the following:

Theorem 6.3. *([22]) There exists a constant $c_n > 0$ such that*

$$\frac{nV(\Omega)}{P(\partial\Omega)} \le \lambda(\Omega) \le c_n V^{1/n}(\Omega).$$

The lower bound is sharp since equality occurs for balls and spherical shells (i.e., a set of the form $\{\mathbf{x} \in \mathbb{R}^n : r < |\mathbf{x} - \mathbf{x_0}| < R\}$).

The constant c_n was calculated explicitly in [22] and is equal to

$$\frac{n^{1+1/n}\Gamma(n/2)\Gamma(\frac{2n-1}{2n-2})^{1-1/n}}{2\pi^{\frac{2n-1}{2n}}\Gamma(\frac{n^2}{2n-2})^{1-1/n}},$$

where, here, Γ is the usual Gamma function. In the case $n = 2$, the quantity $c_n V^{1/n}(\Omega)$ reduces to the sharp estimate $\sqrt{Area(\Omega)/\pi}$. The obstacle in proving the sharpness of the upper bound for $n \ge 3$ comes from the fact that in the Ahlfors-Beurling estimate for the maximum

$$\max_{x\in\Omega} \|\nabla(\int_\Omega \frac{dV(y)}{|x-y|^{n-2}})\|_\infty,$$

the extremal solids are *not* balls in all dimensions ≥ 3, although they are very symmetric algebraic surfaces that are getting more and more tightly sealed to the tangent plane at the maximum point (see [22, p. 82]). The following conjecture proposed in [22] remains open.

Conjecture 6.1. $\lambda(\Omega) \leq R_{vol}$.

The following theorem is the analogue of Theorem 2.2 and gives conditions equivalent to the attainment of the lower bound in Theorem 6.3.

Theorem 6.4. *([22]) TFAE:*
(i) $\lambda(\Omega) = \frac{nV}{P}$.

(ii) There exists $\vec{\varphi} \in B(\Omega)(!) : \vec{x} - \lambda \vec{n}(x) = \vec{\varphi}(x)$ *on* $\partial\Omega$*, where* $\vec{n}$ *is the outward unit normal to* $\partial\Omega$.

(iii) $\frac{1}{V}\int_\Omega u\,dV = \frac{1}{P}\int_{\partial\Omega} u\,d\sigma$ *for all* u *harmonic in* Ω *such that* $\int_S \frac{\partial u}{\partial n} d\sigma = 0$ *for all closed surfaces* S *in* Ω.

(iv) There exists u *in* Ω *satisfying*

$$\Delta u = 1;$$
$$\frac{\partial u}{\partial n}|_{\partial\Omega} = const;$$
$$u|_{\partial\Omega} = local\ constant.$$

The following conjecture thus follows naturally:

Conjecture 6.2. $\lambda(\Omega) = \frac{nV}{P} \Leftrightarrow \Omega$ *is either a ball or a spherical shell.*

Serrin's theorem in higher dimensions implies that if an extremal domain is homeomorphic to a ball, then it must be a ball; however Conjecture 6.2 is still open for domains whose boundary contains more than one component, or domains (such as a torus) that are not homeomorphic to a ball.

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