

Annual Perspectives in Mathematics Education

Using Research to Improve Instruction

2014



NATIONAL COUNCIL OF
TEACHERS OF MATHEMATICS

Annual Perspectives in Mathematics Education

Using Research to Improve Instruction

2014

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NATIONAL COUNCIL OF
TEACHERS OF MATHEMATICS

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CONTENTS

Preface.....	v
PART I Change.....	1
1. Transitioning to Common Mathematics Standards: Computational Fluency in the K–5 Curriculum	3
<i>Dawn Teuscher, Barbara J. Reys, Shannon Dingman, and Amanda Thomas</i>	
2. Implementing a New National Curriculum: A Japanese Public School's Two-Year Lesson-Study Project	13
<i>Akihiko Takahashi and Thomas McDougal</i>	
3. Learning Trajectories for Interpreting the K–8 Common Core State Standards with a Middle-Grades Statistics Example.....	23
<i>Alan P. Maloney, Jere Confrey, Dicky Ng, and Jennifer Nickell</i>	
4. Allowing Students to Take the Lead in Mathematical Investigations.....	35
<i>Lisa B. Warner, Roberta Y. Schorr, and Steven J. Warner</i>	
5. The Effect of Instruction on Developing Autonomous Learners in a College Statistics Class	45
<i>Hope Marchionda, Summer Bateiba, and Melanie Autin</i>	
6. Flipped Classrooms and Task Engagement: Beyond Portable Lectures.....	55
<i>Jeremy F. Strayer and Brandon R. Hanson</i>	
PART II Problem Solving.....	65
7. Creating a Classroom Culture That Encourages Students to Persist on Cognitively Demanding Tasks	67
<i>Doug Clarke, Anne Roche, Peter Sullivan, and Jill Cheeseman</i>	
8. Developing Strategic Competence by Teaching Using the Common Core Mathematical Practices.....	77
<i>Jennifer M. Sub and Padmanabhan Seshaiyer</i>	
9. Increasing Access to Mathematics through Locally Relevant Curriculum.....	89
<i>Janine T. Remillard, Caroline B. Ebby, Vivian Lim, Luke T. Reinke, Nina Hoe, and Emily Magee</i>	
10. Using Rich Tasks to Promote Discourse	97
<i>Denise A. Spangler, JiSun Kim, Dionne Cross, Hulya Kilic, F. Asli Iscimen, and Diana Swanagan</i>	
PART III Reasoning, Explaining, and Discourse	105
11. Supporting Writing with the Student Mathematician Discourse Framework.....	107
<i>Tutita M. Casa</i>	

12. Situating Expansions of Students' Explanations in Discourse Contexts	119
<i>Jeffrey Choppin</i>	
13. Multimodal Communication: Promoting and Revealing Students' Mathematical Thinking	129
<i>Gwendolyn Moffett, Kristen Malzahn, and Mark Driscoll</i>	
14. Teacher Discourse Moves: Supporting Productive and Powerful Discourse	141
<i>Michelle Cirillo, Michael D. Steele, Samuel Otten, Beth A. Herbel-Eisenmann, Kathleen McAney, and Jamila Q. Riser</i>	
15. Promoting Mathematical Reasoning through Critiquing Student Work	151
<i>Catherine Bénéteau, Sarah K. Bleiler, and Denisse R. Thompson</i>	
PART IV Seeing Structure and Generalizing	161
16. Using Classroom Evidence to Inform and Improve Teaching	163
<i>Christine M. Phelps, Felice S. Shore, and Sandy M. Spitzer</i>	
17. Exploring Functions in Elementary School: Leveraging the Representational Context	171
<i>Darrell Earnest</i>	
18. Research-Based Modifications of Elementary School Tasks for Use in Teacher Preparation	181
<i>Jennifer M. Tobias, Dana Olanoff, Amy F. Hillen, Rachael M. Welder, Ziv Feldman, and Eva Thanheiser</i>	
19. Increasing Proficiency Levels of Mathematical Practices.....	193
<i>Janet Hart Frost and Jacqueline Coomes</i>	
PART V Assessment for Teaching and Learning	203
20. Mathematical Practices: Small Changes in Assessments = Big Benefits.....	205
<i>Patricia D. Hunsader, Denisse R. Thompson, and Barbara Zorin</i>	
21. Student Mathematicians Developed through Formative Assessment Cycles	215
<i>Nicole Rigelman and Kellie Petrick</i>	
22. Using Research to Inform Formative Assessment Techniques	229
<i>Michelle Stephan, George McManus, and Robin Dehlinger</i>	
23. Making Mathematical Errors Springboards for Learning	239
<i>Wendy S. Bray and Rossella Santagata</i>	
24. Embedded Field Experiences as Professional Apprenticeships.....	249
<i>Thomas E. Hodges and Heidi Mills</i>	

Preface

Last year, the Seventy-Fifth (and final) Yearbook of the National Council of Teachers of Mathematics (NCTM 2013*) was published as a celebration of the record of rich contributions these annual publications have made to the field of mathematics education. Longtime NCTM members regularly comment on the yearbooks as “a favorite from NCTM,” one prized enough to earn an office shelf dedicated to their personal volumes. (You can learn more about the yearbook’s history in the preface and introduction to the seventy-fifth volume.) Why, then, would NCTM want to change such an important publication?

When the NCTM yearbook was first published in 1926, it was the only book published annually by the Council. Since that time, NCTM has substantially increased the number of its publications and broadened their scope. Thus the NCTM Board of Directors and members recognized a need to reexamine the yearbook’s role and purpose. In 2010, the NCTM Board of Directors appointed a Yearbook Task Force charged with considering the role of the yearbook among the many other NCTM publications (i.e., “Does the Yearbook continue to serve a need for members and others in the field?”) and possibly proposing alternatives to the yearbook. As a result of the task force’s final recommendation, the Board of Directors approved the creation of annual publication designed to uphold the strong traditions of the yearbook while also changing in ways that would reflect twenty-first-century needs and opportunities as well as initiate a more global conversation about mathematics education. As part of this change, the Yearbook Task Force recommended a new title. The title of “yearbook” made sense when the volume was the only book of the year from NCTM. Moreover, NCTM surveys of members revealed that many newer members did not understand the purpose of the yearbook—some members even wondered if it contained photos of members much like a high school yearbook. Thus, the title *Annual Perspectives in Mathematics Education* clearly reflects this book’s role and purpose: providing members with a range of perspectives on timely topics in mathematics education each year. *APME* will take on important topics, offering a range of authors (including classroom teachers, university researchers, professional developers, and occasionally educators outside of mathematics education), targeting a diverse audience that reflects our membership, and providing a collection of chapters that span the pre-K–16 spectrum.

As we considered how *APME* could best meet current needs and take advantage of technological advances in publishing, we decided to institute some different procedures from those used to create the yearbook. In today’s world, research is published at a faster rate, policies change frequently, and educators need access to current and high-quality information, perspectives, and findings much more quickly than in the past. In order to ensure that *APME* is timely, the timeline from conceptualizing the topic or theme to printing needed to be reduced. The topic for this initial 2014 volume—*Using Research to Improve Instruction*—was selected less than two years prior to its publication. This is approximately half of the time that had been allotted for planning,

*National Council of Teachers of Mathematics (NCTM). *Defining Mathematics Education: Presidential Yearbook Selections, 1926–2012*. Seventy-fifth Yearbook of the National Council of Teachers of Mathematics, edited by Francis (Skip) Fennell. Reston, Va.: NCTM, 2013.

selecting chapters for, editing, and printing each yearbook. Moreover, the time from when *APME* authors first submit manuscripts to publication is approximately one year—a considerably shorter publication timeline than authors experience with most professional journals. Given that all *APME* manuscripts undergo a blind peer-review process (at least three reviewers read each manuscript, provide recommendations, and then either ask for revisions or decline the manuscript) and that manuscripts were often substantially revised based on reviewers' and editors' recommendations, *APME*'s timeline is aggressive.

Developments in the publishing process have enabled this ambitious timeline, and dedicated and time-sensitive efforts from authors, editorial panel members, and editors were also critical to this change. For this very first *APME* volume, we experienced an enthusiastic response to a call for manuscripts. Indeed, after undergoing a blind review, the acceptance rate was 27 percent. With the shortened timeline of *APME* and the corresponding opportunity to publish perspectives on issues as they affect educators, students, and schools, it is our hope, and that of NCTM, that readers will find *APME* at the forefront of today's issues. Our goal is to create a publication that addresses current topics, provides high-quality manuscripts from a range of perspectives, and exposes readers to the most current research to inform practice.

Another important element of *APME* relates to access. Online access to professional publications has become an expectation in today's world. Individual chapters from most NCTM yearbooks could not be accessed in the same way as journal articles through library systems or NCTM's online website. When the Yearbook Task Force first identified this problem in 2010, NCTM quickly moved to provide access to individual yearbook chapters with an online purchase option through the NCTM website. However, library access remained difficult. *APME* is classified as a periodical published annually so that library systems will treat each chapter as a journal article. With this change, chapters will be easy to access through library-based searches or through other forms of online searches. An additional benefit of this classification, for those authors needing to report on the impact of their work, is that *APME* now falls under the more traditional academic category of a blind, peer-reviewed journal that is published annually.

■ Creating the Inaugural Volume

As Amy Roth McDuffie (*APME*'s first series editor) and the members of the NCTM's Educational Materials Committee (EMC) looked to select an initial topic and a volume editor, they aimed to create a volume that would be important and immediately beneficial to all NCTM members. For this reason, they decided to focus on the goal that has been at the core of NCTM's mission since its inception: supporting teachers to improve instruction through research-based approaches. Karen Karp was selected to edit this first volume, and she in turn selected Barbara Dougherty, Francis (Skip) Fennell, Elham Kazemi, Matt Larson, Travis Olson, Nelson Palmer, and Christine Suurtamm to round out the Editorial Panel. The Editorial Panel made a purposeful decision to rely strictly on a peer-review process for all *APME* manuscripts and not to extend invitations for particular authors to write chapters on specific topics. The panel then discussed and decided on themes and topics of interest for the volume and collected these in a list. Next, for the first time an Intent to Submit application was used so that the Editorial Panel would be able to plan with these possible topics in mind; fortunately, an enthusiastic response from hundreds of authors was received. There were several levels to the review process with all authors responding under tight deadlines. The chapters were regrouped into five sometimes overlapping clusters

that the panel decided were at the forefront of mathematics education: Change; Problem Solving; Reasoning, Explaining, and Discourse; Seeing Structure and Generalizing; and Assessment for Teaching and Learning. The results are exciting and provocative, and the Editorial Panel is hopeful that readers will agree and find that the 2014 inaugural volume of *APME* makes an important contribution to supporting the teaching and learning of mathematics.

■ In Appreciation

First and foremost, the Editorial Panel members were central to the work herein. They read chapters multiple times, giving detailed comments, edits, and direction, and they responded in a very rapid succession of reviews.

We also would like to acknowledge the contributions and guidance of the NCTM Headquarters staff. Those individuals who supported the production of the volume include Ken Krehbiel, associate executive director for communications; Joanne Hodges, senior director of publications; Myrna Jacobs, publications manager; Larry Shea, copy and production editor; Elizabeth Pontiff, text editor; Kathe Richardson, meeting planner; and many others who worked behind the scenes. In addition, a number of NCTM members provided guidance and input on the volume and on the purpose of *APME*, including William Speer and Rheta Rubenstein (previous general editors of the NCTM yearbook), Susan Garthwaite and Rick Hudson (EMC Chairs during the production of this volume), EMC committee members, Yearbook Task Force members, as well as many other NCTM members who were willing to share their ideas. We are grateful for the long hours, considered attention, and commitment to quality from all involved in bringing this volume to fruition.

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Promoting Mathematical Reasoning through Critiquing Student Work

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Mathematical reasoning, eventually resulting in formal proof, is an essential component of mathematical learning and can range “from informal explanation and justification to formal deduction, as well as inductive observations. Reasoning often begins with explorations, conjectures at various levels, false starts, and partial explanations before a result is reached” (National Council of Teachers of Mathematics [NCTM] 2009, p. 4). Often, reasoning is described as a habit of mind. Indeed, the third Standard for Mathematical Practice of the Common Core State Standards for Mathematics (CCSSM) focuses on the need to “construct viable arguments and critique the reasoning of others,” reinforcing the importance of reasoning throughout the curriculum (National Governors Association Center for Best Practices [NGA Center] and Council of Chief State School Officers [CCSSO] 2010, p. 6).

Instructors at all levels often struggle with how to develop rich, cognitively demanding tasks to engage students in reasoning about mathematics and critiquing mathematical arguments. Preservice secondary mathematics teachers (PSTs), in particular, need experiences within their mathematics content courses with such tasks to facilitate their own mathematical growth and to reflect upon the potential benefit of such tasks for their future high school students. Throughout these experiences, PSTs should become more sophisticated in their reasoning and their “standards for accepting explanations should become more stringent” (NCTM 2000, p. 342).

In this chapter, we share a strategy used with thirty-one PSTs in a mathematics-content course to engage them in discourse with their peers and the instructors. Although our work occurred within the context of geometry, we believe the strategy can be applied across content domains as well as grade levels (e.g., middle or high school courses). We share the strategy and its implementation as well as comments from PSTs that indicate how the activities may have influenced their learning. We relate implications for practice to a framework related to metaknowledge about proof.

■ Context

A group of mathematicians and mathematics educators collaborated to design and teach a geometry course and then a content pedagogy course in the subsequent semester with many of the same PSTs. Within geometry, we had several goals:

- PSTs would learn mathematics through an inquiry approach.
- PSTs would use discourse, often within small groups, to reason about mathematics.
- PSTs would develop proficiency in writing mathematical proofs, and use mathematical language appropriately. (Thompson et al. 2012)

We used Stylianides's (2007) conceptualization of proof to frame our perspective: "Proof is a *mathematical argument*, a connected sequence of assertions for or against a mathematical claim." This sequence incorporates a "*set of accepted statements*" that does not require additional justification and uses "*modes of argumentation*," which are expressed with "*modes of argument representation*" accessible to the classroom community, that is, the students and instructors who jointly discuss the proof (p. 291, italics in original). We believed that Stylianides's conceptualization of proof aligned well with our goals, goals which underscored the importance of the classroom community and public justification of knowledge. We were interested in designing, developing, and adapting strategies that could be used repeatedly throughout the semester and that addressed our goals. Specifically, we used one strategy—critiquing sample student work—in multiple situations to highlight identified misconceptions and provide contexts in which PSTs could engage in rich mathematical discourse.

■ Critiquing Sample Student Work

Critiquing student work made PSTs the mathematical authority in the university classroom, and therefore, as instructors, our role in selecting or creating appropriate sample work to cover a range of responses was crucial. The work we had the PSTs examine included statements or uses of definitions, geometric sketches, mathematical arguments, and formal proofs. We either created sample responses or used PSTs' responses from homework or quizzes to highlight misconceptions. This strategy of examining sample responses empowered PSTs to become their own best critics. Because they expected to evaluate their peers' work, they became more adept at creating and improving their own definitions and proofs of theorems.

When we gave PSTs sample work to evaluate, we attempted to articulate the focus of their critique. If the sample work was in response to a certain question, we expected the PSTs to determine which answer was the most convincing and explain why, paying attention to three criteria: (1) was the response correct (i.e., were there any mathematical inaccuracies?); (2) was the response clear (i.e., did they understand the response?); and (3) was the response complete (i.e., did the work answer the question that had been asked?). Sometimes two possible responses would be correct, but one might be more convincing because it was more complete. Our goal was to have the PSTs identify such differences, because such identification is essential as teachers plan instruction to help all students develop a deep mathematical understanding. In some cases, PSTs modified the sample response so it satisfied the three criteria.

Because one goal was to improve PSTs' ability to understand and write proofs, our critique tasks addressed two areas where PSTs demonstrated misconceptions. The first area focused on the concept of definitions and the second on the need to clearly identify the hypotheses and conclusion of mathematical statements. We viewed both as important precursor knowledge for moving PSTs toward a sophisticated understanding of proof as conceptualized by Stylianides (2007). We share sample responses used for critique related to each of these areas.

Sample Tasks Involving Definitions

Early in the course, we found that PSTs struggled with understanding a certain definition, perhaps because they did not see the need for the mathematical precision of the definition, another critical area for teachers as their students need to address precision as part of CCSSM's mathematical practice 6 (NGA Center and CCSSO 2010). As one example, to have PSTs understand the difference between the measure of an arc in circular degrees and the measure of an arc in length, we had them consider the sample responses and question in figure 15.1.

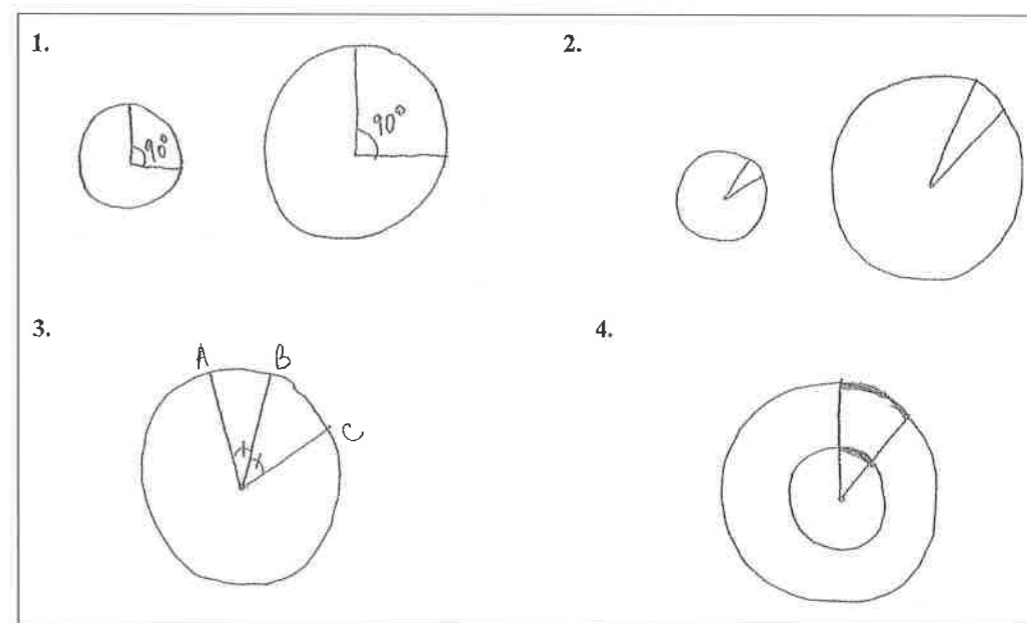


Fig. 15.1. Four student responses to the task, "Draw and label an example of two noncongruent arcs that contain the same number of circular degrees each."

Of the four responses, only response 3 was incorrect (not meeting criteria 1), because the arcs were congruent. Response 2 "looked right" but was not labeled, so it exemplified a response that was not clear (not meeting criteria 2). This particular response is typical of reasoning in geometry, where students think a picture that "looks right" is correct, providing evidence of the perceptual proof scheme (Sowder and Harel 1998). Response 1 was labeled, and therefore was an improvement over response 2, but did not meet criteria 3 for completeness because the angle is not an arbitrary angle. Response 4 was the strongest response (meeting all three criteria); in some sense, it contained the "proof" that the two arcs contained the same number of circular degrees, because

the central angle subtended by the two arcs is the same angle. This response could have been even stronger had the student clearly indicated that the two arcs were not congruent, so completeness is not an absolute standard but falls along a continuum.

Thus, this task forced the PSTs to examine not only mathematical inaccuracies (criteria 1) but different levels of clarity in a visual representation (criteria 2) and the completeness of the response (criteria 3). The PSTs had an opportunity to discuss this task in groups. As an opportunity for individual accountability, we gave the following task on an exam:

- A student in a high school geometry class argues as follows: "Since the measure of the inscribed angle BAC is $1/2$ of the arc BC , and since the measure of the inscribed angle DAE is $1/2$ of the arc DE , and since angle BAC is congruent to angle DAE , arcs BC and DE are congruent." (See fig. 15.2.) Explain which parts of the student's answer are correct and what the student's error in thinking is.

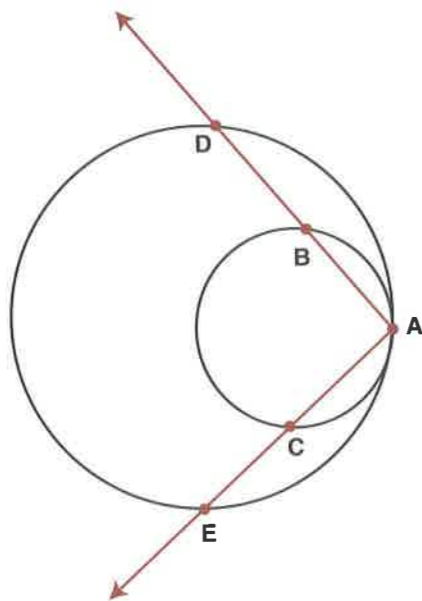


Fig. 15.2. Figure for congruent arc problem

In this hypothetical argument, the student concluded incorrectly that the two arcs BC and DE are congruent; the steps in the hypothetical argument are correct until the last line, when the student confused the measure of the arc in circular degrees with the measure of the arc length. The PSTs needed to recognize that the hypothetical student did not apply the correct context for the definition of arc measure and confused arc length with number of circular degrees.

Sample Tasks Involving the Identification of Hypotheses and Conclusions

As we began instruction on proof, we realized that PSTs struggled with identifying the hypotheses and conclusions of statements. Sometimes, as instructors, we spend considerable time trying to communicate sophisticated ideas relative to proof development, such as ideas related to appropriate modes of argumentation (e.g., direct or indirect reasoning) or modes of argument representation (e.g., symbolic or narrative), when the basic concepts of the initial assumptions and the conclusion are not clear to our students. As a preliminary task relative to this issue, PSTs examined sample responses in which the theorem "Bisectors of supplementary angles are perpendicular" had been written in if-then form. Indeed, the statement of the theorem appears to some PSTs to reflect mathematical "truth," rather than a statement that has a hypothesis and a conclusion. Identifying the assumption that is implicit in the phrasing of the statement was a challenging task for many.

Again, PSTs evaluated sample responses based on the criteria of correctness, clarity, and completeness. Sample responses from a previous quiz included:

1. If supplementary angles have bisectors, then they are perpendicular.
2. If you bisect supplementary angles, then the bisector is perpendicular.
3. If you can construct bisectors of supplementary angles in a figure, then those bisectors are perpendicular.
4. If two angles are supplementary, then their bisectors are perpendicular.

Notice that the first three responses did not pinpoint that the assumption of the statement is that the angles are supplementary. In addition, language issues naturally arose. For example, in response 1, it is unclear whether the pronoun "they" refers to the bisectors or the angles. In response 2, the student is unable to make meaning of the words "bisect" and "bisector" and so wrote a sentence with no meaning; in essence, the student considered a bisector to be perpendicular on its own without reference to another line.

At a more advanced level, PSTs were given the task in figure 15.3. Most of the proof is correct, except for the first step. This error occurs because the student used the "fact" that two angles are "alternate interior angles" to conclude that the angles are congruent. In other words, the response used the assumption that two of the sides of the quadrilateral are parallel to conclude that the quadrilateral is a parallelogram; the relatively subtle error implicitly used the conclusion instead of the hypothesis.

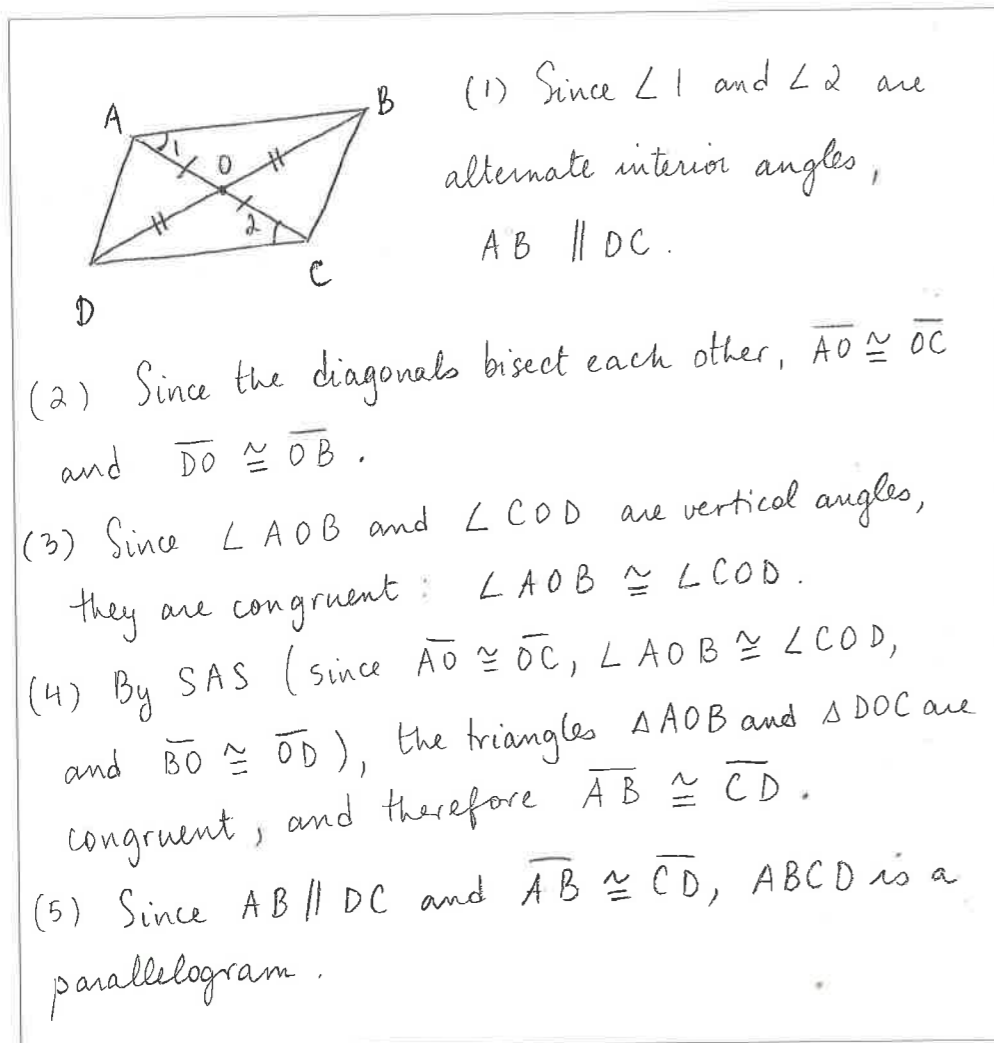


Fig. 15.3. Student response to the task, "Find, and if possible, fix the errors in the proof of the following statement: If the diagonals of a quadrilateral bisect each other, then the quadrilateral is a parallelogram."

■ Implementing Critiques of Sample Work

We used varied implementation approaches to engage PSTs with critiquing sample student work. We previously described two approaches: (1) having PSTs discuss sample responses in groups and then share with the class and (2) having PSTs evaluate and validate sample responses on an exam. Here we describe two additional implementation approaches (i.e., sharing via chart paper and random reporting) and discuss a major issue we faced.

Chart paper was a useful medium for sharing. For instance, PSTs worked in groups, with each group creating a Frayer model for a definition (Frayer, Frederick, and Klausmeier 1969) or generating a sample argument or proof for a statement. We posted their work on chart paper around

the room, without names. In groups, the PSTs circulated to read the samples from their peers and attached a group grade to each sample on sticky notes. Then, during whole-class sharing, PSTs justified why they gave a response a particular grade, providing PSTs an opportunity to engage in an activity that will become a regular part of their careers, namely evaluating and validating student work. Hearing rationales provided by others for why a response received a particular grade helped everyone become more critical and discerning in reading responses. Thus, the PSTs again took on the role of the mathematical authority in the university class rather than the instructors.

We often had the PSTs work collaboratively to critique sample responses and wanted to ensure that all group members participated. We had a serendipitous finding from sharing via random reporting (Kagan 1992). We numbered the groups and then the individuals within the group. When we were ready to share, we rolled a die to identify the group and rolled the die again to identify the individual in the group who would respond. This small change in our sharing had a major impact on group dynamics from the standpoint of the PSTs. Because they did not know who would be the spokesperson for the group, they all felt the need to ensure that every group member was able to speak for the group. Once this change in individual accountability was implemented, random reporting became the standard means of sharing from groups to the entire class.

A major issue we faced was building a body of knowledge, especially in a proof-based course such as geometry, when many of the PSTs had taken geometry in high school and had previous knowledge (with or without proof) of the material. To address the uncertainty of what information was acceptable to use, we introduced the idea of the "toolbox" and what was in it. As the course progressed, definitions and theorems that had been proved in class were added to the toolbox and could be used to construct future proofs (compare "set of accepted statements" [Stylianides 2007, p. 291]). The use of the toolbox was a metaphor that clarified and made explicit for some PSTs what assumptions they were allowed to make when attempting to prove a given statement. The toolbox metaphor is one that secondary teachers can use with their own students who will have seen many of the concepts in geometry and algebra in earlier grades but often without justification.

■ Implications for Practice: Critiquing Sample Work to Focus on Metaknowledge about Proof

We have discussed how we used critiquing hypothetical and actual student work (within different implementation structures) as a primary means of having PSTs reflect on two areas of difficulty, namely, the concept of definition and the need to clearly identify the hypotheses and conclusion of a given mathematical statement. At the end of the geometry course, we surveyed the thirty-one PSTs about their perceptions of proof. This was to help us understand how our strategies related to proof instruction affected PSTs' perceptions of their own proof ability and how we might refine our approach to teaching reasoning and proof in the future. Here we share what we learned from those surveys and from a further reading of research literature. We share implications for practice related to using the strategy of critiquing sample work to focus more explicitly on metaknowledge about proof.

One survey question was, "What does it mean to prove something?" Many of the PSTs responded to this question with statements such as, "To prove something means to provide factual information for a generalized concept" or "To show that something always works." These responses suggested that PSTs may have been thinking about proofs as demonstrating validity in a vacuum without consideration of the idea that mathematicians prove an implication within a larger theory

dependent on axioms and prior knowledge that has been deduced in that system. Less frequent were responses from PSTs that indicated attention to the idea that a mathematical proof proves an implication (If A, then B) rather than proving a fact (Cabassut et al. 2012). An exemplary, but uncommon, model of such a response was, “You show that if A then B or if A is true then B. You start with A then use your toolbox to prove B.”

The responses from our PSTs caused us to reflect on how we might modify our approach to more explicitly point to the idea that proof is not a “stand-alone concept” (Balacheff 2009, p. 118) but depends on and exists within a larger theory (Cabassut et al. 2012), a central tenet of mathematical sense making. Mathematics teachers at the K–12 and university levels often assume that, through engaging in proof construction and related activities, individuals will obtain a metalevel understanding of the role and meaning of proof within the discipline. However, students at all levels persist in their difficulties with proof concepts that may be ameliorated through explicit attention to the notion of proof, what Cabassut and colleagues (2012) refer to as “metaknowledge about proof” (p. 181). More important, it is precisely this metaknowledge about proof that is essential for teachers if they are to provide authentic experiences for their own students in proof and argumentation and an accurate portrayal of mathematics as a discipline.

Cabassut and colleagues (2012) suggested the use of “mini-theories” to help individuals think about this metaknowledge. The toolbox metaphor might be a useful step in developing mini-theories, in that PSTs came to understand that the validity of their arguments depended on the set of tools (axioms, previously proven theorems, etc.) that were accepted in their community (see Stylianides 2007). Similar to the mini-theories proposed by Cabassut and colleagues, we hoped the toolbox would allow PSTs to gain a sense of the structure and systemization of mathematics. When reflecting on activities they believed helped them become proficient with proof, several mentioned the toolbox as facilitating their understanding of proof concepts. For example, one PST wrote, “At the beginning of most lectures, the class breaks up into groups and investigates certain characteristics/properties of geometric shapes that will lead up to the proofs. After each point we are able to put the ‘knowledge’ into our ‘toolbox’ to use for later proofs of theorems.” We believe PSTs recognized the toolbox as a norm in the class, but may not have made connections to the metalevel understanding that all mathematical proofs exist and are valid only within a theoretical construct, such as the axioms and theorems of Euclidean geometry.

According to Cabassut and colleagues (2012), few ideas have been shared about how to incorporate metaknowledge about proof into classroom instruction. Thus, an implication for practice from our work is that critiquing of sample work with PSTs has the potential for leading to explicit discussions on Cabassut’s components of metaknowledge about proof. For example, one component is related to an understanding of “the absolute certainty of mathematics,” which “resides not in the facts but in the logical inferences, which are often implicit” (p. 183). Critiquing related to identifying hypotheses and conclusions can serve as a jumping-off point for explicit discussion related to the role of implication in proof. Another component of metaknowledge about proof is “the conscious use of definitions” that must “be understood and applied in their exact meanings” (p. 185). Critiquing of tasks, such as recognizing that different conclusions follow from two distinct definitions for measure of an arc, could highlight the importance of precision in the use of definitions while engaging PSTs in mathematical reasoning.

In constructing sample responses, we addressed misconceptions that arose in class related to the role and nature of proof; these provided opportunities for PSTs to reflect on these issues.

However, their reflections cannot be left implicit; in future instruction, we plan to incorporate a greater focus on metaknowledge about proof through explicit discussions resulting from their critiques of sample work.

■ Conclusion

We have described a global strategy, namely critiquing sample student work, with various implementations, for courses at any grade level in which inquiry and discourse are goals of instruction, even if enhancing proof-writing proficiency is not a goal. For K–12 students, this strategy supports CCSSM’s Standard for Mathematical Practice 3, which includes “critique the reasoning of others” (NGA Center and CCSSO 2010, p. 6); for PSTs, using this strategy provides a model for future instruction. An added benefit of using the strategy was removing the teacher as the authority; the PSTs themselves, often working in groups, became the authorities. Not only did the PSTs become engaged and responsible for their own learning, their critical thinking and mathematical proof-writing skills also improved. The strategy engaged PSTs as learners of mathematics and provided practice in evaluating and validating student work, something they will need to do regularly as teachers. Their survey comments suggested that the toolbox idea, the collaborative work in critiquing arguments, and the proof construction and critique as a whole class enhanced their own proof proficiency. We believe the strategy we described and the various implementation approaches (sharing in groups and then as a class, via chart paper and random reporting; evaluating and validating sample work on exams) are applicable in many contexts beyond geometry to help build deep conceptual understanding of mathematical ideas. If PSTs experience powerful strategies in their own learning, we believe they will want to incorporate such strategies into their future instruction and be able to help their students develop metaknowledge about proof and its role in the discipline of mathematics.

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■ PART IV

Seeing Structure and Generalizing