SELECTED PROBLEMS IN CLASSICAL FUNCTION THEORY

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Abstract. We discuss several problems in classical complex analysis that might appeal to graduate students and young researchers. Among them are possible extensions to multiply connected domains of the Neuwirth-Newman theorem regarding analytic functions with positive boundary values, characterizing domains by properties of best approximations of $\overline{z}$ by analytic functions in various metrics, and sharpening the celebrated Putnam inequality in the context of Toeplitz operators on Bergman spaces and the related isoperimetric inequalities, aka " isoperimetric sandwiches".

1. Introduction: Spaces of Analytic Functions

This paper is a selective survey of a few problems that are at the interface of complex analysis and geometry. We will be dealing with various classes of analytic functions in arbitrary domains, such as Hardy, Bergman, and Smirnov spaces. Let us begin by defining these spaces (see [8, 9, 16]).

Definition 1.1. *For $0 < p < \infty$, define the Bergman space of the disk to be*

$$A^p(\mathbb{D}) = \left\{ f \text{ analytic in } \mathbb{D} : \left(\int_{\mathbb{D}} |f(z)|^p dA(z) \right)^{\frac{1}{p}} =: \|f\|_p < \infty \right\},$$

where $dA = \frac{1}{\pi} dxdy$ denotes normalized area measure in the unit disk $\mathbb{D}$. The Bergman spaces $A^p(G)$ for an arbitrary domain G are defined in a similar way.

If instead of area measure, we consider line integrals on concentric circles, we get the Hardy spaces.

Definition 1.2. *For $0 < p < \infty$, define the Hardy space of the disk as*

$$H^p(\mathbb{D}) := \left\{ f \text{ analytic in } \mathbb{D} : \sup_{0<r<1} \frac{1}{2\pi} \int_0^{2\pi} |f(re^{it})|^p dt =: \|f\|_{H^p}^p < \infty \right\}.$$

When $p = \infty$, we define

$$H^\infty(\mathbb{D}) = \{ f \text{ analytic in } \mathbb{D} : \sup \{ |f(z)|, z \in \mathbb{D} \} =: \|f\|_\infty < \infty \}.$$

For arbitrary domains G, we define the Hardy spaces as follows (see [11]).

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Definition 1.3. *An analytic function $f(z)$ in G belongs to the Hardy class $H^p(G)$ for $0 < p < \infty$ if the subharmonic function $|f|^p$ has a harmonic majorant in G.*

Note that Hardy classes are conformally equivalent, i.e., if $\varphi : K \to G$ is the conformal mapping of an n-connected circular domain K onto G, then $f \in H^p(G)$ if and only if $f \circ \varphi \in H^p(K)$. On the other hand, if one defines a space in a way analogous to that of Definition 1.2 for an arbitrary domain G, one gets a potentially different class of functions called *Smirnov* classes ([8, 14]). More specifically, let G be an n-connected domain in the complex plane bounded by Jordan rectifiable curves γ_1,...,γ_n and let $\Gamma = \bigcup_{i=1}^n \gamma_i$.

Definition 1.4. *Let $0 < p < \infty$. An analytic function $f(z)$ in G is said to belong to the Smirnov class $E^p(G)$ if there exists a sequence of rectifiable curves $\{\Gamma_i\}$ in G converging to Γ such that*

$$\limsup_{i\to\infty} \int_{\Gamma_i} |f(z)|^p |dz| =: \|f\|^p_{E^p} < \infty.$$

Notice that the critical difference between Smirnov classes and Hardy classes is that the former are not conformally equivalent. In addition, for $p \geq 1$, Hardy functions are represented by Poisson integrals of their boundary values, while Smirnov functions are represented by Cauchy integrals, i.e., in the former case the kernel is positive, in the latter, complex. This difference will allow for some interesting phenomena related to boundary values, which we discuss in the next section. In Section 3, we examine the concept of *analytic content*, which is the best approximation of the simplest anti-analytic function, namely $\bar{z}$, in an appropriate context. We will see that analytic content is tightly connected to the geometry of a domain. In Section 4, we discuss Putnam's inequality for Toeplitz operators on Bergman spaces. In each section, we state conjectures and mention some open problems.

2. Analytic Functions with Positive Boundary Values

For $p \geq 1$, it is well-known that functions in $H^p(\mathbb{D})$ cannot have real boundary values on the circle unless they are constants. This is because Hardy space functions ($p \geq 1$) can be represented as Poisson integrals of their boundary values, and therefore if they are real on the boundary, they are real in the whole domain, and thus must be constant. However, for $0 < p < 1$, there are many such functions, for example, $f(z) = i\frac{1+z}{1-z}$. A beautiful theorem of J. Neuwirth and D. J. Newman ([24]) shows that if we require the boundary values to be positive, we can go a little further:

Theorem 2.1. *([24]) Let $f \in H^{1/2}(\mathbb{D})$ be such that $f(\zeta) \geq 0$ for almost every ζ on the unit circle $\mathbb{T}$. Then f is constant.*

Neuwirth and Newman pointed out that the value $1/2$ in the theorem is sharp, since the function $\frac{z}{(1+z)^2}$ is in $H^p(\mathbb{D})$ for all $0 < p < 1/2$ and has positive boundary values. If we consider arbitrary domains G that might have "bad boundaries", we can replace non-tangential boundary values with asymptotic boundary values (see [19]), and interpret "almost everywhere" as being understood with respect to harmonic measure, and then the theorem extends to $H^p(G)$.

An interesting question is, what happens for the Smirnov classes $E^p(G)$ when $p \geq 1$? The answer depends on the geometric character of the boundary. Recall that a finitely connected domain G is called Smirnov if the derivative of the conformal map from a circular domain onto G is an outer function (see [8]). It turns out that if the domain G is non-Smirnov, then the theorem is false, and indeed in such a domain, for every $0 < p < \infty$, there exist non-constant $f \in E^p(G)$ such that $0 \leq f(\zeta) \leq 1$ for almost every ζ on the boundary of the domain (see [18]). In the other extreme, if the domain G has smooth boundary Γ, then the classes $H^p(G)$ and $E^p(G)$ are equal (as sets), and therefore the situation in $E^p(G)$ is exactly the same as that of $H^p(G)$. However, if the domain is Smirnov and has singularities, these singularities allow for the construction of functions in E^p with real boundary values, for certain values of p. In fact, the values of p that allow for the construction of such functions are tightly connected in general to the geometric characteristic of the singularity. For more detail on the construction of such functions with real boundary values, see [6, 7] and the references therein.

In the case that there do exist non-trivial functions in E^p with real boundary values, if G is a simply connected Smirnov domain, L. DeCastro and D. Khavinson noted that the analogue of the Neuwirth-Newman theorem holds:

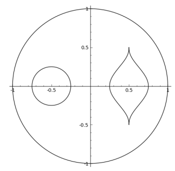
Theorem 2.2. ([7]) *Let G be a simply connected Smirnov domain with rectifiable boundary Γ. Let $p_0 \geq 1$ be defined as the smallest $p \geq 1$ such that $f \in E^p(G)$ and f has real boundary values a.e. on Γ imply that f is a constant. Then all $f \in E^{p_0/2}$ such that $f \geq 0$ a.e. on Γ are constants.*

The proof of this theorem is along the same lines as that of the original Neuwirth-Newman result, and is sketched here.

Proof. Write $f(z) = B(z)S(z)F^2(z)$, where $B(z)$ is a generalized Blaschke product, $S(z)$ is a bounded singular inner function, and $F(z) \in E^{p_0}$ is an outer function. On Γ, since $f \geq 0$, we have that $B(z)S(z)F^2(z) = |f(z)|$. On the other hand, $|f(z)| = |F(z)|^2 = F(z)\overline{F(z)}$ a.e., and therefore $\overline{F(z)} = B(z)S(z)F(z) \in E^{p_0}(G)$. This implies that $F(z) + \overline{F(z)} \in E^{p_0}(G)$ and is real-valued, hence a constant. Thus, $f(z) = const \cdot B(z)S(z)$ is a bounded function with non-negative boundary values, hence a constant as well. □

In multiply connected domains, it is not clear whether the Neuwirth-Newman theorem holds. We can still write $f = QBSF^2$, where B is the generalized Blaschke product, S is a singular inner function, F^2 is an outer factor, $F \in E^{p_0}$, Q is an invertible bounded analytic function, and $|Q|$ is a local constant on Γ (see [19]). The problem is that $|B|$ and $|S|$ are local constants a.e. on the boundary of G. Hence, $f \geq 0$ a.e. on Γ only yields on Γ that $f = QBSF^2 = |QBS|F\overline{F}$, i.e., $\overline{F}$ coincides with *different* analytic functions on different boundary components.

However, L. DeCastro and D. Khavinson showed (see [7]) that in an n - connected domain G of "cardioid type" (as pictured below) with m interior cusps on ∂G, all E^1-functions with positive boundary values are constants. (For such domains, $p_0 = 2$.)



It seems likely that the proof of this result will hold for finitely connected domains with finitely many corners, and thus, the authors of that paper conjectured that the Neuwirth-Newman theorem holds in multiply connected domains:

Conjecture 2.1. *([7]) Let G be a finitely connected Smirnov domain. If $p_0 \geq 1$ is the smallest index for which all functions in E^{p_0} with real boundary values are constants, then all $E^{p_0/2}$ functions with positive boundary values are constants.*

3. Analytic Content

Let us now turn to a discussion of the approximation of $\bar{z}$ by analytic functions from different classes in domains with analytic boundaries. The main focus will be the concept of *analytic content*.

3.1. **Bounded Functions.** Let G be a finitely connected region in $\mathbb{C}$ with boundary Γ consisting of n simple closed analytic curves γ_j, $j = 1, \ldots, n$.

Definition 3.1. *The analytic content of a domain G is*

$$\lambda(G) := \inf_{\phi \in H^\infty(G)} \| \bar{z} - \phi \|_{L^\infty(\Gamma)} .$$

We often call ϕ the *best approximation* of $\bar{z}$ in $H^\infty(G)$. The analytic content turns out to have an interesting relationship with certain geometric features of the domain, as can be seen by the following theorem.

Theorem 3.1. *([2, 21, 13]) Let G be a finitely connected region whose boundary is analytic, and let A and P be the area and perimeter of G, respectively. Then*

$$\frac{2A}{P} \leq \lambda(G) \leq \sqrt{\frac{A}{\pi}}.$$

Moreover, $\lambda(G) = \sqrt{\frac{A}{\pi}}$ if and only if G is a disk.

Note that by ignoring the analytic content in the inequality stated above, one recovers the classical isoperimetric inequality, namely that $P^2 \geq 4\pi A$. In addition, the theorem states that the upper bound is achieved if and only if G is a disk, and therefore a natural question is, for which G does the equality $\frac{2A}{P} = \lambda(G)$ hold? The following theorem gives some equivalent forms for the achievement of the lower bound.

Theorem 3.2. *(*[21]*) Let G be a finitely connected region whose boundary is analytic, and let A and P be the area and perimeter of G. The following are equivalent:*

(i) $\lambda = \frac{2A}{P}$;

(ii)There is $\phi \in H^\infty(G)$ *such that* $\bar{z}(s) - i\lambda \frac{d\bar{z}}{ds} = \phi(z(s))$ *on* Γ, *where s is the arc-length parameter;*

(iii) $\frac{1}{A}\int_G f dA = \frac{1}{P}\int_\Gamma f ds$ *for all* $f \in H^\infty(G)$.

Notice that (iii) holds for annuli $G = \{r < |z| < R\}$! Therefore, the lower bound does indeed hold for regions other than disks. However, D. Khavinson proved in [21] that if the domain is simply connected, then it is a disk. On the other hand, if the domain is finitely connected, then (ii) implies that if Γ contains a circular arc, G is a disk or an annulus. In [21], the author asked whether these are the only two possibilities. It turns out that the answer is yes! This has been recently proved in [1].

3.2. **Smirnov Classes.** If we consider the Smirnov classes $E^p(G)$ instead of bounded analytic functions in G, we can generalize the notion of analytic content. In what follows, $L^p(\Gamma) := L^p(\Gamma, ds)$, where s is the arc length parameter.

Definition 3.2. *For* $p \geq 1$, *the Smirnov analytic content of a domain G is*

$$\lambda_{E^p}(G) := \inf_{\phi \in E^p(G)} \parallel \bar{z} - \phi \parallel_{L^p(\Gamma)} .$$

This extremal quantity turns out to be equal to another, often referred to as the "dual extremal problem":

$$\sup_{f \in E_1^q(G)} \left| \int_\Gamma \bar{z} f(z) dz \right| .$$

Here, $E_1^q(G)$ refers to the unit ball in $E^q(G)$, where $1/p + 1/q = 1$. One might ask if there are upper and lower bounds for the analytic content for the Smirnov classes similar to those in Theorem 3.1, and indeed there are.

Theorem 3.3. *(*[15]*) Let A, P denote the area and perimeter of a finitely connected domain G. For* $p \geq 1$, $q = \frac{p}{p-1}$, *we have*

$$\frac{2A}{\sqrt[q]{P}} \leq \lambda_{E^p} \leq \sqrt{\frac{A}{\pi}} P^{\frac{1}{p}}.$$

Again, one might ask, are disks and annuli the only extremal domains for all λ_{E^p}, $p \geq 1$? In addition, do the extremal functions for λ_{E^p} characterize the domain G? The following theorem gives some insight into the second question.

Theorem 3.4. *(*[15]*) Let* $\Gamma := \partial G$ *be real analytic and* $p \geq 1$. *If the best approximation to* $\bar{z}$ *in* E^p *is a constant, then G is a disk.*

Sketch of the proof: Without loss of generality, let's assume that the best approximation is zero. Then one can show that $0 \in G$ and that the extremal function f^* for the dual problem satisfies $|f^*| \leq 1$ in G and $|f^*| = 1$ on the boundary, and the duality relationship

$$f^* \bar{z} dz = const|z|^p ds \quad \text{on } \Gamma$$

holds, where we can take the constant to be positive. Dividing by z yields

$$\frac{f^*(z)}{z} dz = const|z|^{p-2} ds. \tag{3.1}$$

For $p = 1$, using regularity results for extremals (see [23]) in order to apply the argument principle, if f^* is not constant, one can show that the left hand side of (3.1) has a non-trivial increment of its argument, while the right hand side doesn't (because it's positive), which is a contradiction. Therefore, we conclude that f^* is a unimodular constant. Now again using regularity of the boundary and parametrizing $z = r(\theta)e^{i\theta}$, and using the duality relationship (3.1) gives, after some simple calculus, that $dr/d\theta = 0$, and hence Γ consists of circles centered at the origin. Using the duality equation one last time shows that since dz/ds must have the same sign on both circles, then there can only be one circle, and hence, G is a disk. The case $p > 1$ is more complicated, and in particular, the case that $p \in \mathbb{N}$ has to be treated separately. For details, see [15].

Note that this theorem *proves* that the domain is simply connected. If we assume G to be simply connected to begin with, the regularity hypothesis (that is, the analyticity of the boundary) can be relaxed significantly to assume merely that G is a Smirnov domain, by appealing to the following theorem.

Theorem 3.5. *([10]) Let G be a Jordan domain in $\mathbb{C}$ containing 0 and with the rectifiable boundary Γ satisfying the Smirnov condition. Suppose the harmonic measure on Γ with respect to the origin equals $c|z|^\alpha ds$ for $z \in \Gamma$, where ds denotes arclength measure on Γ, $\alpha \in R$ and c is a positive constant. Then*

(i) For $\alpha = -2$, the solutions are precisely all disks G containing 0.

(ii) For $\alpha = -3, -4, -5, \dots$ there are solutions G which are not disks.

(iii) For all other values of α, the only solutions are disks centered at 0.

The conclusion of Theorem 3.4 then follows, because the left hand side of (3.1) is a constant multiple of harmonic measure at the origin, and since in our case, $\alpha = p - 2$ with $p > 1$ so $\alpha > -1$, part (iii) of Theorem 3.5 applies, giving that G is a disk.

What happens in the finitely connected case is not known, and thus leads to the following problem.

Problem 3.1. *Extend Theorem 3.4 to finitely connected Smirnov domains. In particular, do the hypotheses of that theorem imply that G is simply connected?*

Notice that in the case $0 < p < 1$, we can still define analytic content, but we lose duality (since in that case E^p is not a Banach space), and so it is not clear whether you might get a type of "duality equation" on the boundary holding for the extremal. Thus one might consider the following.

Problem 3.2. *What can be said about analytic content in E^p spaces for $0 < p < 1$? Are there estimates similar to those in Theorem 3.3?*

In a similar manner as before, one might ask whether best approximations of $\bar{z}$ characterize annuli. The following theorem gives the answer when $p = 1$.

Theorem 3.6. *([15]) Let $\Gamma := \partial G$ be real analytic and $p = 1$. If the best approximation to $\bar{z}$ in E^1 is a rational function $g(z) = \frac{c}{z-a}$, then G is an annulus centered at a.*

Conjecture 3.1. *Theorem 3.6 holds for all $p > 1$ and all finitely connected Smirnov domains.*

The following problem is completely unknown territory, and it is easy to see that the study of such domains leads to a larger class than the well-known quadrature domains.

Problem 3.3. *Study domains where best approximations of $\bar{z}$ in E^p are, say, rational functions.*

3.3. **Bergman Spaces.** Let us now discuss analytic content in the context of Bergman spaces.

Definition 3.3. *For $p \geq 1$, the Bergman space analytic content of a domain G is*

$$\lambda_{A^p}(G) := \inf_{\phi \in A^p(G)} \| \bar{z} - \phi \|_{A^p(G)} .$$

Theorem 3.7. *([15]) Let G be a Smirnov domain and let $p \geq 1$. Then*

(i) If the best approximation of $\bar{z}$ in A^p is a constant, then G is a disk.

(ii) If the best approximation of $\bar{z}$ in A^p is $g(z) = \frac{c}{z-a}$, then G is an annulus centered at a.

Sketch of proof. For (i), assume for the sake of brevity that $p > 1$ and that the best approximation of $\bar{z}$ is 0. Recall that Khavin's lemma (see [27]) states that the annihilator $(A^p)^{\perp}$ of A^p inside $L^q(G, dA)$ is given by the $\bar{z}$ derivatives of functions in the standard Sobolev space $W_0^{1,q}$, where p and q are conjugate indices. Therefore, the dual problem in this context states that

$$\inf_{\phi \in A^p(G)} \| \bar{z} - \phi \|_{A^p(G)} = \sup_{u \in W_0^{1,q}, \|u_{\bar{z}}\|_q \leq 1} \left| \int_G \bar{z} \frac{\partial u}{\partial \bar{z}} \, dA(z) \right| .$$

The duality relationship then yields that for $z \in G$,

$$\frac{\partial u}{\partial \bar{z}} = const \frac{|z|^p}{\bar{z}}, u \in W_0^{1,q}$$

where u is a solution of the dual problem. Integrating with respect to $\bar{z}$ gives (in G):

$$u = const|z|^p + h, \text{ where } h \in H^\infty(G).$$

Since $u = 0$ on Γ, we get that $h|_\Gamma$ is real-valued and therefore constant on Γ, and therefore $|z|$ is constant on Γ, hence Γ is a disk. For the proof of (ii), see [15].

Note that this proof is easier in the context of Bergman spaces, because the duality relationship holds in the whole domain G.

Problem 3.4. *What are the isoperimetric "sandwich" estimates for λ_{A^p}?*

Nothing is known about the following problem.

Problem 3.5. *What can be said about domains with other rational best approximations of $\bar{z}$ in A^p? For example, if the best approximation is a rational function of degree 2, what is the corresponding domain?*

4. Putnam's Inequality for Toeplitz Operators in Bergman Spaces

Let us now turn to a discussion of isoperimetric inequalities in the context of operator theory. Recall that if T is a bounded linear operator on a Hilbert space, then T is called hyponormal if $[T^*, T] := T^*T - TT^* \geq 0$. Putnam's inequality (see [26]) applied to T then states that

$$\|[T^*, T]\| \leq \frac{Area(sp(T))}{\pi}$$

where $sp(T)$ denotes the spectrum of T. In particular, if ϕ is analytic in a neighborhood of the finitely connected domain G and $T := T_\phi : E^2 \to E^2$ is defined by $Tf = \phi f$, (T is called an analytic Toeplitz operator), then Putnam's inequality in this context states that

$$\|[T^*, T]\| \leq \frac{Area(\phi(G))}{\pi}.$$

In [20], the author gave a lower bound:

$$\frac{4(Area(\phi(G)))^2}{\|\phi'\|^2_{E^2(G)} \cdot P} \leq \|[T^*, T]\|,$$

where $\partial(\phi(G)) =: \Gamma$, and $P := P(\Gamma) =$ perimeter of $\phi(G)$. Putting the above two inequalities together and taking $\phi(z) = z$ gives $P^2 \geq 4\pi A$, the classical isoperimetric inequality. If $\phi(z) = z$ and $G = \phi(G) = \mathbb{D}$, then equality is achieved, and thus, Putnam's inequality in the E^2 context is sharp in the sense that there exists an operator on E^2 for which Putnam's inequality becomes equality.

One might then ask what happens in spaces other than E^2. The authors in [12] explore this question for Bergman spaces, following the paper [3]. Without loss of generality, we can assume there exists a measure μ in $\mathbb{C}$ such that T is unitarily equivalent to the operator T_z of multiplication by z on $L^2_a(\mu)$, the closure (in $L^2(\mu)$) of functions analytic in a neighborhood of the support of μ

(see [3]). Letting K be the support of the measure μ and G the polynomial hull of K, a standard Hilbert space calculation then shows that

$$\|[T_z^*, T_z]\| = \sup_{\|g\|_2=1} \left\{ \inf_{f\in H^\infty(G)} \|\bar{z}g - f\|_2^2 \right\}.$$

Taking $f = gh$ then gives this right hand side less than or equal to

$$\inf_{h\in H^\infty(G)} \|\bar{z} - h\|_\infty^2 =: \lambda^2(G).$$

Here $\lambda := \lambda(G)$ is the analytic content for bounded functions from Definition 3.1. Thus, Theorem 3.1 and Putnam's inequality give that $\|[T_z^*, T_z]\| \le \lambda^2 \le A(G)/\pi$, and therefore equality is attained in Putnam's inequality only if the spectrum $sp(T)$ is a disk and the spectral measure "sits" on the circumference. Thus it is clear that in the context of Bergman spaces, for example, equality can never be attained in Putnam's inequality. A calculation (straightforward but tedious!) reveals that in $A^2(\mathbb{D})$, $\|[T_z^*, T_z]\| = 1/2$, that is, the upper bound is two times smaller than the one in the general Putnam inequality. One might ask, then, should Putnam's inequality in this context be corrected by a factor 1/2?

In exploring this question, the authors of [4] considered the torsional rigidity ρ of a domain G, which measures the resilience of the beam of cross section G to twisting. In terms of a "Rayleigh type" quotient,

$$\rho := \sup_{\psi\in C_0^\infty} \left(\frac{2\|\psi\|_1}{\|\nabla\psi\|_2} \right)^2.$$

They then proved the following.

Theorem 4.1. *([4])*

$$\frac{\rho}{Area(\phi(G))} \le \|[T_\phi^*, T_\phi]\|.$$

Hence, taking $\phi = z$ and using the upper bound given by Putnam's inequality gives the "isoperimetric sandwich"

$$\rho \le \frac{(Area(G))^2}{\pi}.$$

The estimate in the above theorem $\rho \le \frac{(\mathrm{Area}(G))^2}{\pi}$ was missing by a factor of 2 the celebrated Saint-Venant inequality conjectured in 1856, which was first proved by G. Polya in 1948. This prompted the following conjecture.

Conjecture 4.1. *([4]) For the Bergman space, $\|[T_z^*, T_z]\| \le \frac{Area(G)}{2\pi}$.*

For simply connected domains G, this conjecture is now a theorem! (See [25].) Hence, this leads to a new proof of Saint-Venant's Inequality that $\rho \le \frac{(\mathrm{Area}(G))^2}{2\pi}$. The proof in [25] is tour de force calculation with power series. This is why the statement is restricted to simply connected domains. The authors of [12] noted that refining Olsen and Reguera's proof implies that the equality for the self-commutator upper bound in simply connected domains holds only for disks.

This yields an alternative proof that Saint-Venant's inequality becomes equality only for disks. We are thus left with a host of interesting problems to investigate.

Problem 4.1. *Find the "book" proof of the Olsen - Reguera theorem in* [25], *freeing it from the power series calculation and extending the result to arbitrary domains.*

Problem 4.2. *Is the sharp upper bound for the A^2-content equal to* $\frac{1}{2}\sqrt{\frac{Area(G)}{\pi}}$*?*

Problem 4.3. *What is the sharp lower bound for the A^2-content expressed in terms of geometric characteristics (e.g., area, perimeter, principal frequency) of the domain?*

Problem 4.4. *Refine the "isoperimetric sandwich" inequalities for $\|[T^*, T]\|$ to include the connectivity of the domain.*

This last problem is virtually unexplored territory. In his thesis in the 70s ([17]), S. Jacobs refined Carleman's celebrated inequality ([5]) bounding the A^2 norm of G in terms of the E^1 norm of G for multiply connected domains. In [22], there is a result connecting geometric characteristics of the domain G (area, perimeter, connectivity, and analytic content) with the mapping properties of φ, the best approximation of $\bar{z}$, and the mapping properties of the extremal function in the dual problem.

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